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Asymptotic Behaviour of Solutions of Linear Recurrences and Sequences of Möbius-Transformations

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This paper is mainly concerned with the study of recurrences defined by Möbius-transformations, whose solutions are the orbits of points on the Riemann-sphere under a sequence of Möbius-transformations. We study the asymptotic behaviour of such solutions in relation to the asymptotic behaviour of the coefficients of the Möbius-transformations. Most of the theorems give sufficient conditions in order that there exist converging solutions, but a section of examples is added where examples are given of recurrences whose solutions do not converge because one or several of the conditions of the theorems are violated. One of the most important results of this paper is that if the fixpoints of the Möbius-transformations are of bounded variation and converge to distinct limits, then the behaviour of the solutions depends entirely on the products of the derivatives in the fixpoints. Several methods will be proposed to deal with the case that the fixpoints converge to one single limit. The paper starts with a few results on nth order recurrences and matrix recurrences and concludes with an investigation of the asymptotic behaviour of the solutions of linear second-order recurrences having coefficients that are asymptotic expressions in fractional powers of the index n. A number of examples are added in order to show how some of the theorems can be applied.

INTRODUCTION AND NOTATION

This paper is divided into ten sections. Each section begins with a discussion of the contents of that section, so the reader who wants to have a quick idea of what this paper is about is advised to skim through the first few lines of each section. The remainder of this introduction serves only to introduce some basic facts and notations that will be needed throughout the paper, for quick reference.

First of all, the solutions of the different types of recurrences that are the subject of this paper are sequences of numbers, vectors, or matrices $\{x_n\}_{n \geqslant n_0}^{\infty}$. Because we are only concerned with the asymptotic behaviour of

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the solutions, the precise value of the starting index is irrelevant, and solutions will be given in the form $\{x_n\}$, leaving out the domain of definition of the indices. If it is necessary to specify such a domain, e.g., to indicate hat a certain identity holds for all n, this will mostly be indicated by $n \in \mathbb{N}$. A similar remark holds for sums and products, which are generally written $\sum_{n=0}^{\infty}(\cdots)$, etc. In a few cases, where an expression involves an infinite sum, where the indices over which the sum extends depend on the convergence of the sum, we shall avail ourselves of the notation $\sum_{(n)}$, which is $\sum_{k=n}^{\infty}$ if the sum converges, and $\sum_{k=0}^{n-1}$ if the sum does not converge (see especially Theorem 1.4).

A matrix recurrence is given by a sequence

$$M_n x_n = x_{n+1} \qquad (n \in \mathbb{N}, n \geqslant N) \tag{1.1}$$

with $\{M_n\}$ a sequence of non-singular $k \times k$-matrices with entries in a field K, where $K = \mathbb{C}$ or $\mathbb{R}$ and $\{x_n\}$ is a sequence of $k \times l$-matrices with rank l ($l \leqslant k$). We call the sequence $\{x_n\}$ an l-dimensional solution of the matrix recurrence.

In addition to matrix recurrences we also study linear (kth order) recurrences, which are given by an equation (or rather, a sequence of equations)

$$u_{n+k} + P_{k-1}(n)\, u_{n+k-1} + \cdots + P_0(n)\, u_n = 0 \tag{1.2}$$

for $n \in \mathbb{N}$, $n \geqslant N$, where the $P_j(n)$ are given sequences of numbers in the field K and $P_0(n) \neq 0$ for $n \geqslant N$. In this case, $\{u_n\}_{n \geqslant N}$, with the numbers $u_n \in K$, is a solution of (1.2), and the sequence $\{(u_{n+k-1}, ..., u_n)^t\}_{n \geqslant N}$ (where t denotes the transpose of a vector or matrix) is a (1-dimensional) solution of the so-called associated matrix recurrence, defined by the Kronecker-matrices

$$M_n = \begin{pmatrix} -P_{k-1}(n) & \cdots & \cdots & -P_0(n) \\ 1 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{1.3}$$

A k-dimensional solution of this associated matrix recurrence is given by a sequence of matrices whose column vectors are of the form $(u_{n+k-1}^{(i)}, ..., u_n^{(i)})^t$ with $\{u_n^{(1)}\}, ..., \{u_n^{(k)}\}$ a complete set of linearly independent solutions of the linear recurrence (1.2). In this way it is possible to translate results about matrix recurrences into results about linear recurrences.

Finally, we give some notation concerning matrices and linear recurrences. The notation I_k will be used to denote the identity matrix in $K^{k,k}$. If there is no ambiguity, we shall often write I instead of I_k. We denote the ith unit vector in K^k by e_i, and for a matrix M, we let M_{ij} be the entry in the ith row and the jth column. If $x \in K^k$ is a vector, x^t denotes the transpose of x.

If the entries of a sequence of matrices converge to a limit matrix M, the characteristic polynomial $\chi(M) = \det(XI - M)$ will, by abuse of language, also be called the characteristic polynomial of the corresponding recurrence. Similarly for a linear recurrence (1.2): if the coefficients $P_j(n)$ converge to numbers P_j as $n \to \infty$, the characteristic polynomial of the recurrence will be $\chi(X) = X^k + P_{k-1} X^{k-1} + \cdots + P_0$. Of course, if (1.1) is the associated matrix recurrence, the characteristic polynomials of (1.1) and (1.2) will be equal.

For a matrix $M \in K^{k,l}$ the norm $\|M\|$ is defined as the matrix norm induced by some vector norm on K^l:

$$\|M\| = \max_{x \neq 0} |Mx|/|x|.$$

By diag($R_1, R_2, ..., R_m$) we denote the (block-)diagonal matrix

$$\begin{pmatrix} R_1 & & & 0 \\ & R_2 & & \\ 0 & & \ddots & \\ & & & R_m \end{pmatrix},$$

where the R_i are square matrices or just numbers in K.

In Theorems 1.1 and 1.2 we use the notation $\mathcal{M}$ to denote the set of functions $F: \mathbb{N} \to \mathbb{R}_{>0}$ such that $f(n)/f(m)$ is bounded from above for all $n > m \geqslant n_0$ and such that $\lim_{n \to \infty} f(n+1)/f(n) = 1$. As an example, $f(n) = 1$ and $F(n) = n^{-a}(\log n)^b$ with $a, b \in \mathbb{R}$, $a > 0$ belong to $\mathcal{M}$.

Lastly, we adopt the following convention: if $\prod_{n=n_0}^{\infty} \lambda_n = 0$ (or ∞) for numbers $\lambda_n \in K$, this will imply that the products $\prod_{n=m}^{p} |\lambda_n|$ are bounded from above (below) for all $p \geqslant m \geqslant n_0$. We shall see that this is a natural convention (e.g., compare Example 9.1).

1. MATRIX RECURRENCES AND *N*TH ORDER RECURRENCES

Although the bulk of this paper is concerned with sequences of Möbius-transformations, we shall avail ourselves of the opportunity to state and prove a few results on matrix recurrences and nth order linear recurrences. In the remainder of the paper these will only be used for recurrences of order two, but since the results are valid for any order, we may as well state them in a more general context. The first result (Theorem 1.2) is an interesting generalization of the well-known Poincaré–Perron Theorem ([13, 15]; for its statement see the remark after the proof of Theorem 1.2) for linear recurrences of order $n > 1$. It states that if the coefficients of a linear recurrence converge and the characteristic polynomial has l zeros whose moduli are equal to a certain number A, then the recurrence has a basis of l solutions that satisfy a linear recurrence of order l with characteristic polynomial $(X-A)^l$. The two other theorems of the first section are, first, a useful result that states to what extent the solutions of a matrix recurrence whose matrices are almost diagonal resemble the solutions of a matrix recurrence whose matrices are diagonal matrices (Theorem 1.4). The proof of this theorem will be given in Section 2. The first application of this result on almost-diagonal matrices is the third theorem of the first section (Corollary 1.6) which states that if the zeros of the characteristic polynomial are distinct, and the coefficients are of bounded variation, then the Poincaré–Perron Theorem is true for this case too (provided that some minor additional condition is met). We conclude the first section with another example that shows how Theorem 1.4 can be applied to matrix recurrences whose coefficients converge fast.

We proceed to the first theorem.

THEOREM 1.1. *Let* $M = \operatorname{diag}(R_1, R_2, ..., R_L)$, *with* $R_j \in K^{k_j, k_j}$ *such that all complex eigenvalues of* R_j *have smaller moduli than those of* R_{j+1} $(j = 1, ..., L-1)$. *Let* $f \in \mathscr{M}$ *such that* $\lim_{n\to\infty} f(n) \in \mathbb{R}$ *and* $\{M_n\}$ *a sequence of matrices in* $K^{k,k}$ *such that*

$$\|M_n - M\| = o(f(n)) \qquad (n \to \infty).$$

Then there exists a sequence $\{G_n\}$ *of non-singular matrices in* $K^{k,k}$ *such that*

$$G_{n+1}^{-1} M_n G_n = \operatorname{diag}(R_{1n}, R_{2n}, ..., R_{Ln})$$

with $\lim_{n\to\infty} R_{jn} = R_j$ *and*

$$\|R_{jn} - R'_{jn}\| = o(\|M_n - M\|) \qquad (n \to \infty;\ j = 1, ..., L),$$

where R'_{jn} *is the submatrix of* M_n *composed of the same rows and columns as* R_j *in* M. *Further*

$$\lim_{n\to\infty} G_n = I$$

and

$$\|G_n - I\| = o(f(n)) \qquad (n \to \infty).$$

Moreover, if $\sum_{n=N}^{\infty} 1/f(n) \cdot \|M_n - M\|$ *converges, then* $\{G_n\}$ *can be found such that* $\sum_{n=N}^{\infty} 1/f(n) \cdot \|G_n - I\|$ *converges as well.*

Proof. See Theorem 3.1 of [6].

Here follows the corresponding result for linear recurrences.

THEOREM 1.2. *Consider the linear recurrence given by* (1.2) *with characteristic polynomial* $\chi(X)=(X-\alpha_1)\cdots(X-\alpha_k)$, *where* $\alpha_1, ..., \alpha_k \in \mathbb{C}$, $|\alpha_1|=\cdots=|\alpha_l|$ *and* $|\alpha_1|\neq|\alpha_j|$ *for* $l<j\leqslant k$. *Let* $f\in\mathcal{M}$ *such that* $\lim_{n\to\infty} f(n)\in\mathbb{R}$ *and* $P_j-P_j(n)=o(f(n))$ $(n\to\infty; j=0, ..., k-1)$. *Then* (1.2) *has l linearly independent solutions* $\{u_n^{(i)}\}$ *such that*

$$u_{n+l}^{(i)}+b_{l-1}(n)\,u_{n+k-1}^{(i)}+\cdots+b_0(n)\,u_n^{(i)}=0 \qquad (i=1, ..., l), \tag{1.4}$$

where $b_j(n)\in K$ $(n\in\mathbb{N})$ *and* $b_j-b_j(n)=o(f(n))$ $(n\to\infty;\ j=0, ..., l-1)$ *for numbers* $b_0, ..., b_{l-1}\in K$ *defined by*

$$X^l+\sum_{j=0}^{l-1} b_j X^j=(X-\alpha_1)\cdots(X-\alpha_l).$$

Moreover, if $\sum_{j=0}^{k-1}\sum_{n=1}^{\infty} 1/f(n)\cdot|P_j(n)-P_j|$ *converges, then* $\sum_{j=0}^{l-1}\sum_{n=1}^{\infty} 1/f(n)\cdot|b_j(n)-b_j|$ *converges as well.*

Proof. Put $\varepsilon(n)=\max_{m\geqslant n}\max_{0\leqslant j\leqslant k-1}|P_j-P_j(m)|$ for $n\in\mathbb{N}$ and let the associated matrix recurrence be given by (1.3). There exists some matrix $V\in K^{k,k}$, V, non-singular, such that $V^{-1}MV=\operatorname{diag}(R, S)$ where $M=\lim M_n$ and R and $S\in K^{l,l}$ have eigenvalues $\alpha_1, ..., \alpha_l$ and $\alpha_{l+1}, ..., \alpha_k$, respectively in $\mathbb{C}$. It follows from Theorem 1.1 that there exists a sequence $\{G_n\}$, $G_n\in K^{k,k}$, with

$$\|G_n-I\|=o(f(n)) \qquad (n\to\infty)$$

and

$$\|G_{n+1}^{-1}V^{-1}M_nVG_n-V^{-1}MV\|=o(\varepsilon(n)) \qquad (n\to\infty)$$

such that

$$G_{n+1}^{-1}V^{-1}M_nVG_n=\operatorname{diag}(R_n, S_n) \qquad (n\in\mathbb{N}),$$

where $\lim R_n=R$, $\lim S_n=S$. Let $Y_n\in K^{l,l}$ be such that $R_nY_n=Y_{n+1}$ and $\det Y_n\neq 0$ and let $X_n=(Y_n, 0)^t\in K^{k,l}$ for all n. Then $\{U_n\}=\{VG_nX_n\}$ is an l-dimensional solution of (1.1). Let $\chi(X)=X^l+\sum_{j=0}^{l-1} b_jX^j$ be the characteristic polynomial of R. Clearly, $\chi\in K[X]$ and $\chi(R)=0$ by the Cayley–Hamilton Theorem. Hence,

$$\begin{aligned}\chi(M)\,U_n &= V\cdot \operatorname{diag}(\chi(R),\chi(S))\cdot G_n\cdot X_n\\ &= W\cdot\chi(S)\cdot\delta(n)\cdot X_n\\ &= W\cdot\chi(S)\cdot\delta(n)\cdot G_n^{-1}\cdot V^{-1}\cdot U_n\\ &= D_n U_n \qquad (n\in\mathbb{N}),\end{aligned} \tag{1.5}$$

where W is the matrix composed of the last $k-l$ columns of V, $\delta(n)\in K^{k-l,k}$ is the matrix composed of the last $k-l$ rows of $G_n - I$ (in fact, the last $k-l$ columns of $\delta(n)$ are irrelevant and can even be taken zero), and $D_n\in K^{k,k}$, $\|D_n\| = O(\|G_n - I\|)$ $(n\in\mathbb{N})$. Equation (1.5) implies that

$$\sum_{j=0}^{l} b_j U_{n+j} + E_n U_n = 0, \tag{1.6}$$

where $b_l = 1$, $E_n\in K^{k,k}$, $\|E_n\| = O(\varepsilon(n) + \|G_n - I\|)$ for $n\to\infty$. On evaluating the lower $k-l$ rows of (1.6), we obtain $k-l$ equations for l linearly independent solutions $\{u_n^{(i)}\}$ of (1.2)

$$\begin{gathered}\sum_{j=0}^{l} b_j u_{n+j}^{(i)} + \sum_{j=1}^{k} (E_n)_{kj} u_{n+k-j}^{(i)} = 0\\ \vdots\\ \sum_{j=0}^{l} b_j u_{n+k-l-1+j}^{(i)} + \sum_{j=1}^{k} (E_n)_{l+1,\,j} u_{n+k-j}^{(i)} = 0.\end{gathered}$$

Since $b_l = 1$, the last equation enables us to express $u_{n+k-1}^{(i)}$ as a linear combination of $u_n^{(i)}, ..., u_{n+k-2}^{(i)}$ with bounded coefficients, which do not depend on i. Substituting the expression for $u_{n+k-1}^{(i)}$ into the first $k-l-1$ equations, we obtain $k-l-1$ equations

$$\begin{gathered}\sum_{j=0}^{l} b_j u_{n+j}^{(i)} + \sum_{j=2}^{k} (E'_n)_{kj} u_{n+k-j}^{(i)} = 0\\ \vdots\\ \sum_{j=0}^{l} b_j u_{n+k-l-2+j}^{(i)} + \sum_{j=2}^{k} (E'_n)_{l+2,\,j} u_{n+k-j}^{(i)} = 0,\end{gathered}$$

where $\|E'_n\| = O(\varepsilon(n) + \|G_n - I\|)$. We can repeat the above procedure, using the last of the $k-l-1$ equations in order to obtain an expression for $u_{n+k-2}^{(i)}$

as a linear combination of $u_n^{(i)}, ..., u_{n+k-3}^{(i)}$ with bounded coefficients. Repeating this procedure until only one equation is left we find that

$$\sum_{j=0}^{l} b_j u_{n+j}^{(i)} + \sum_{j=k-l}^{k} (E_n^{(k-l-1)})_{kj}\, u_{n+k-j}^{(i)} = 0$$

for $i = 1, ..., l$ and

$$\|E_n^{(k-l-1)}\| < c \cdot (\varepsilon(n) + \|G_n - I\|) = o(f(n)) \qquad (n \to \infty)$$

for some constant c. Q.E.D

In particular, if we have $l = 1$, in other words, if the characteristic polynomial has an eigenvalue α with multiplicity one and such that all other eigenvalues have moduli $\neq |\alpha|$, Theorem 1.2 states that there exists a solution $\{u_n\}$ of (1.2) with

$$u_{n+1} - (\alpha + \delta(n))\, u_n = 0$$

with $\delta(n) = o(f(n))$ for $n \to \infty$, so that $u_{n+1}/u_n - \alpha = o(f(n))$ $(n \to \infty)$. If, in addition, $\sum_{n=1}^{\infty} 1/f(n) \cdot |P_i(n) - P_i|$ converges for $i = 0, ..., k-1$, then, by Theorem 1.2, $\sum_{n=0}^{\infty} 1/f(n) \cdot |\delta(n)|$ converges as well. Using that $f \in \mathscr{M}$, we then have that $\sum_{k=n}^{\infty} |\delta(k)| < c \cdot \sum_{k=n}^{\infty} (|\delta(k)|/f(k)) \cdot f(n) = o(f(n))$ for some constant c. Hence, if $\alpha \neq 0$, then

$$u_n = u_0 \alpha^n \prod_{h=0}^{n-1} (1 + \delta(h)/\alpha),$$

so that u_n/α^n converges and, for a suitable choice of u_0,

$$\frac{u_n}{\alpha^n} - 1 = o(f(n)) \qquad (n \to \infty)$$

and, for $\alpha = 0$, it follows that

$$\sum_{n=1}^{\infty} \frac{1}{f(n)} \cdot \left| \frac{u_{n+1}}{u_n} \right| < \infty.$$

Remark. If all zeros $\alpha_1, ..., \alpha_k$ of χ have distinct moduli, then there is a basis of solutions $\{u_n^{(i)}\}$ of (1.2) $(i = 1, ..., k)$ such that for all i, $\lim_{n \to \infty} (u_{n+1}^{(i)}/u_n^{(i)}) = \alpha_i$. This is the original Poincaré–Perron Theorem (see [13, 15] for the original papers and see, for the corresponding matrix version [5, 6] or [9]). Moreover, in [3] Gelfond and Kubenskaya proved that if in (1.2), $P_j - P_j(n) = O(\beta(n))$ for some real function β with $\lim_{n \to \infty} \beta(n+1)/\beta(n) = 1$ and $\sum_{n=0}^{\infty} \beta(n) < \infty$, and the characteristic polynomial χ has zeros $\alpha_1, ..., \alpha_k$ which are nonzero and have distinct moduli, then there are solutions $\{u_n^{(i)}\}$

with $u_n^{(i)} = \alpha_i^n(1 + O(\sum_{h=n}^{\infty} \beta(n)))$. We show that this follows from Theorem 1.2 as well: clearly $\beta \in \mathscr{M}$. Let $f \in \mathscr{M}$ such that $\beta(n) = o(f(n))$. We have just seen that there exists, for every zero α of χ such that $|\alpha| \neq |\alpha'|$ for α' any other zero of χ, some solution $\{u_n\}$ of (1.2) such that $u_{n+1}/u_n - \alpha = o(f(n))$. Since f is arbitrary, we must have $u_{n+1}/u_n - \alpha = O(\beta(n))$. Hence $u_n = C \cdot \alpha^n \prod_{h=0}^{n-1} (1 + O(\beta(h)))$ for some constant C. Since $\prod_{h=0}^{\infty} (1 + O(\beta(h)))$ converges, we have, for a suitable value of C,

$$\begin{aligned} u_n &= \alpha^n \prod_{h=n}^{\infty} (1 + O(\beta(h))) \\ &= \alpha^n \left(1 + O\left(\sum_{h=n}^{\infty} \beta(h)\right)\right) \qquad (n \in \mathbb{N}). \end{aligned}$$

In fact, the result we get is somewhat stronger: if α is a zero of χ such that the modulus of α is distinct from the moduli of the other zeros of χ, then there is a solution $u_n = \alpha^n(1 + O(\sum_{h=n}^{\infty} \beta(h)))$.

Theorem 1.2 in a sense reduces the case that the linear recurrence (1.2) has converging coefficients to the case that all eigenvalues of the characteristic polynomial have equal moduli.

Subsequently we state and prove a result for the case that all zeros of the characteristic polynomial are distinct. On the one hand, this condition is weaker than the condition in the Poincaré–Perron Theorem, which requires that all zeros have distinct moduli. On the other hand, we have to impose additional conditions on the coefficients, because in general it is not true in this case that a basis $\{u_n^{(1)}\}, ..., \{u_n^{(k)}\}$ of solutions exists such that $\lim_{n \to \infty}(u_{n+1}/u_n) = \alpha_i$ for all zeros α_i (see Example 9.3 and Section 10). In fact, we require only that the coefficients of the recurrence are of bounded variation (this condition was also used in work on orthogonal polynomials, e.g., [10, 11, 17]). We first prove the result in a somewhat more general setting, i.e., for matrix recurrences.

THEOREM 1.3. *Let $\{M_n\}$ be a sequence of non-singular $k \times k$-matrices with coefficients in the field $K = \mathbb{R}$ or $\mathbb{C}$, such that $M = \lim_{n \to \infty} M_n$ exists and has eigenvalues $\alpha_1, ..., \alpha_k$ which are all distinct and such that the products of quotients*

$$\prod_{h=m}^{M} \left| \frac{\alpha_i(h)}{\alpha_{i+1}(h)} \right| \tag{1.7}$$

are bounded from above for all m, M and $i = 1, ..., k-1$, where $\alpha_1(n), ..., \alpha_k(n)$ are the eigenvalues of M_n that converge to $\alpha_1, ..., \alpha_k$, respectively. Further *suppose that* $\sum_n \|M_n - M_{n+1}\|$ *converges. Then there exists a sequence* $\{G_n\}$, $G_n \in K^{k,k}$, *such that* $\lim_{n\to\infty} G_n = G$, $\det G \neq 0$, *and*

$$G_{n+1}^{-1} M_n G_n = \operatorname{diag}(\alpha_1(n), ..., \alpha_k(n)) \qquad (n \in \mathbb{N})$$

if neither of the α_i *is zero, and*

$$G_{n+1}^{-1} M_n G_n = \operatorname{diag}(\alpha_1(n) + \delta_n, ..., \alpha_k(n)) \qquad (n \in \mathbb{N})$$

if $\alpha_1 = 0$, *for numbers* $\delta_n = O(\|M_n - M_{n+1}\|)$.

For the proof we use the following auxiliary results:

THEOREM 1.4. *Let* $\{A_n\} = \{\operatorname{diag}(a_1(n), ..., a_k(n))\}$ *be a sequence of diagonal non-singular matrices in* $K^{k,k}$ *such that for all m and p large enough the quotients* $\prod_{h=m}^{p} |a_i(h)/a_j(h)|$ *are bounded from above for all* $i < j$ *and let* $\{D_n\}$ *be a sequence of matrices in* $K^{k,k}$ *such that* $A_n + D_n$ *is non-singular for all n and* $\sum_{n=0}^{\infty} (\|D_n\|/|a_j(n)|) < \infty$ *for all* $j > L \geqslant 0$, $j \leqslant k$. *Then there exists a sequence* $\{G_n\}$, $G_n \in K^{k,k}$ *with*

$$\lim_{n\to\infty} G_n = I_k$$

and

$$G_{n+1}^{-1} \cdot (A_n + D_n) \cdot G_n = \operatorname{diag}(P_n + Z_n R_n, a_{L+1}(n), ..., a_k(n)) \qquad (n \in \mathbb{N}), \tag{1.8}$$

where $P_n \in K^{L,L}$ *is the matrix that consists of the first L rows and columns of* $A_n + D_n$, $R_n \in K^{k-L,L}$ *is the matrix that consists of the last* $k - L$ *rows and the first L columns of* $A_n + D_n$, *and where* Z_n *is a* $L \times (k-L)$ *matrix with*

$$\|Z_n\| \leqslant C' \cdot \sum_{h=0}^{n} \frac{\|D_h\|}{|a_{L+1}(h)|} \prod_{j=h+1}^{n} \left|\frac{a_L(j)}{a_{L+1}(j)}\right|$$
$$= \frac{\|D_n\|}{|a_{L+1}(n)|} O(1) + \left|\frac{a_L(n)}{a_{L+1}(n)}\right| O(1)$$

for some constant C'. *Furthermore,*

$$\|G_n - I\| \leqslant C \cdot \max_{i,j} \prod_{h=0}^{n-1} \left|\frac{a_i(h)}{a_j(h)}\right| \cdot \sum_{(n)} \frac{\|D_l\|}{|a_{L+1}(l)|} \prod_{q=0}^{l} \left|\frac{a_j(q)}{a_i(q)}\right| \tag{1.9}$$

for some constant C independent of $n \in \mathbb{N}$, *where the maximum is taken over all pairs* (i, j) *such that at least one of the* i, j *is greater than L.*

Proof. See Section 2.

In this paper, we shall apply Theorem 1.4 only for $L=0$ or $L=1$, as in Corollary 1.6 and in Theorem 7.1. After Corollary 1.6, at the end of this section, we shall apply Theorem 1.4 to the case that the matrices A_n are constant.

LEMMA 1.5. *Let $\chi_n(X)=\sum_{j=0}^{k} P_j(n)\, X^j$ be a sequence of monic polynomials converging to $\chi(X)=(X-\alpha_1)\cdots(X-\alpha_k)$, where $\alpha_1, ..., \alpha_k$ are pairwise distinct complex numbers. If $\sum_{n=0}^{\infty} |P_j(n)-P_j(n+1)|$ converges for $j=0, ..., k-1$, then there exist k sequences of complex numbers $\alpha_i(n)$ such that $\chi_n(\alpha_i(n))=0$, $\lim_{n\to\infty} \alpha_i(n)=\alpha_i$ and $\sum_{n=1}^{\alpha} |\alpha_i(n)-\alpha_i(n+1)|$ converges ($n\in\mathbb{N}$, $i=1, ..., k$).*

Proof. Since the zeros of a monic polynomial depend continuously on the coefficients we see that for n large enough, χ_n has zeros $\alpha_1(n), ..., \alpha_k(n)$ such that

$$|\alpha_i-\alpha_i(n)|<\tfrac{1}{2}\min_{i\neq j} |\alpha_i-\alpha_j|.$$

We show that $\sum_{n=0}^{\infty} |\alpha_i(n)-\alpha_i(n+1)|$ converges for all i. Fix $i\in\{1, ..., k\}$. Clearly, $\chi_n(\alpha_i(n))=0$ ($n\in\mathbb{N}$), we have

$$\alpha_i(n+1)-\alpha_i(n)=\frac{\chi_n(\alpha_i(n+1))-\chi_{n+1}(\alpha_i(n+1))}{\int_0^1 \chi_n'(t\alpha_i(n+1)+(1-t)\,\alpha_i(n))\,dt}.$$

Since χ is monic and $\chi'(\alpha_i)\neq 0$, the denominator is bounded from below for n large enough and it follows that $\sum_{n=0}^{\infty} |\alpha_i(n)-\alpha_i(n+1)|<\infty$. In particular, $\alpha_i(n)$ converges and, by the choice of $\alpha_i(n)$, the limit can only be α_i. Q.E.D

Proof of Theorem 1.3. Let $\chi_n(X)=X^k+P_{k-1}(n)\, X^{k-1}+\cdots+P_0(n)$ be the characteristic polynomial of M_n. Since the coefficients $P_j(n)$ lie in the ring generated by the entries of M_n, we have $\sum_{n=0}^{\infty} |P_j(n)-P_j(n+1)|<\infty$ for $0\leqslant j\leqslant k-1$. By Lemma 1.5, we conclude that χ_n has zeros $\alpha_1(n), ..., \alpha_k(n)$ such that $\sum_{n=0}^{\infty} |\alpha_i(n)-\alpha_i(n+1)|$ converges and $\lim_{n\to\infty} \alpha_i(n)=\alpha_i$ ($i=1, ..., k$). Define sequences of eigenvectors $f_1(n), ..., f_k(n)$ of M_n such that $M_n f_i(n)=\alpha_i(n)\, f_i(n)$ and such that $f_i(n)$ converges to f_i, an eigenvector of M with eigenvalue α_i. In fact, because the rank of $M_n-\alpha_i(n)\, I_k$ is $k-1$ there is some $\lim_{n\to\infty} \mu_{lj}(n)\neq 0$, where $\mu_{lj}(n)$ is the cofactor of $(M_n-\alpha_i(n)\, I_k)_{lj}$. We can now take

$$f_i(n)=(\mu_{lj}(n))^{-1}\,(\mu_{l1}(n), ..., \mu_{lk}(n))^t.$$

Then it is clear that $\sum_{n=0}^{\infty} |f_i(n)-f_i(n+1)|$ converges.

Put $\hat{G}_n = (f_1(n), ..., f_k(n))$. Then $\lim \hat{G}_n = G$, with G a matrix of eigenvectors for M and $\hat{G}_n^{-1} M_n \hat{G}_n = \operatorname{diag}(\alpha_1(n), ..., \alpha_k(n))$. Moreover,

$$\sum_{n=0}^{\infty} \|\hat{G}_{n+1}^{-1} \hat{G}_n - I\| < c \cdot \sum_{n=0}^{\infty} \|\hat{G}_{n+1} - \hat{G}_n\| < \infty,$$

for some constant c, so that $\hat{G}_{n+1}^{-1} M_n \hat{G}_n = \operatorname{diag}(\alpha_1(n), ..., \alpha_k(n)) + D_n$, where $\|D_n\| = O(\|M_n - M_{n+1}\|)$ whence $\sum_{n=0}^{\infty} \|D_n\| < \infty$. Application of Theorem 1.4 (for $L = 0$) immediately yields the desired result. Q.E.D

COROLLARY 1.6. *Consider the linear recurrence* (1.2). *Suppose that* $\sum_{n=0}^{\infty} |P_j(n) - P_j(n+1)|$ *converges for* $j = 0, ..., k-1$, *and that the zeros* $\alpha_1(n), ..., \alpha_k(n)$ *of the characteristic polynomials* $\chi_n(X) = X^k + P_{k-1}(n) X^{k-1} + \cdots + P_0(n)$ *converge to distinct complex numbers* $\alpha_1, ..., \alpha_k$, *and that the quotients* (1.7) *are bounded from above. Then there exists a basis of solutions* $\{u_n^{(1)}\}, ..., \{u_n^{(k)}\}$ *of* (1.2) *such that for* $i = 1, ..., k$ *if* $\alpha_i \neq 0$,

$$u_n^{(i)} = (1 + o(1)) \prod_{h=0}^{n-1} \alpha_i(h) \qquad (n \in \mathbb{N}),$$

and if $\alpha_i = 0$, *then*

$$\frac{u_{n+1}^{(i)}}{u_n^{(i)}} = (\alpha_i(n) + O(d(n)))(1 + o(1)),$$

where $d(n) = \sum_{j=0}^{k-1} |P_j(n) - P_j(n+1)|$.

Proof. Consider the associated matrix recurrence $M_n x_n = x_{n+1}$, where the M_n are given by (1.3). By Theorem 1.3, there exists a sequence of matrices $\{U_n\}$, such that

$$U_{n+1}^{-1} M_n U_n = \operatorname{diag}(\alpha_1(n), ..., \alpha_k(n))$$

if none of the α_i are zero, whereas if, say, $\alpha_1 = 0$, then

$$U_{n+1}^{-1} M_n U_n = \operatorname{diag}(\alpha_1(n) + O(d(n)), ..., \alpha_k(n))$$

and where U_n converges to U, a matrix of eigenvectors for M. One-dimensional solutions of the associated matrix recurrence are of the form $(u_{n+k-1}, ..., u_n)^t$, with $\{u_n\}$ a solution of (1.2). For all i, we set $y_n^{(i)} = (\prod_{h=0}^{n-1} (U_{h+1}^{-1} M_h U_h)_{ii})\, e_i$, where e_i is the ith unit vector. If $\alpha_i \neq 0$, then

$y_n^{(i)} = (\prod_{h=0}^{n-1} \alpha_i(h))\, e_i$, and if $\alpha_i = 0$, then $y_n^{(i)} = (\prod_{h=0}^{n-1} (\alpha_i(h) + O(d(h))))\, e_i$. Clearly, $\{U_n y_n^{(i)}\}$ is a solution of the associated matrix recurrence. Further,

$$U_n e_i = c_i(1 + o(1)) \cdot (\alpha_i^{k-1}, ..., \alpha_i, 1)^t \qquad (i = 1, ..., k)$$

for some non-zero constant c_i. This completes the argument. Q.E.D

Condition (1.7), which figures in both Theorem 1.4 and Corollary 1.6, cannot be dispensed with, as will be shown in Example 9.1. On the other hand, in practice, it is almost always satisfied (for an example, see Section 10).

Before concluding this section on matrix recurrences, we shall, as an example, show how Theorem 1.4 can be applied to obtain the following result of Evgrafov [2].

Consider a linear recurrence (1.2) *with* $\sum_{n=0}^{\infty} |P_j(n) - P_j| < \infty$, *where* $P_j = \lim_{n\to\infty} P_j(n)$. *If the characteristic polynomial has zeros* $\alpha_1, ..., \alpha_k$ *with* $0 < |\alpha_1| \leqslant \cdots \leqslant |a_k|$, *then* (1.2) *has solutions* $u_n^{(i)} = \alpha_i^n(1 + o(1))$.

One of the reasons we give this example is to show how formula (1.9) applies. In fact, we prove even more, giving an estimate for the order of convergence of $u_n^{(i)}/\alpha_i^n$, in the same way as we did above (see the remark after Theorem 1.2).

PROPOSITION 1.7. *Let* $\alpha_1, ..., \alpha_k$ *be non-zero, not necessarily distinct numbers with* $|\alpha_1| \leqslant \cdots \leqslant |\alpha_k|$ *and let* $\beta\colon \mathbb{N} \to \mathbb{R}_{>0}$ *be a function such that* $\lim_{n\to\infty} \beta(n) = 0$, $\sum_{n=0}^{\infty} \beta(n) < \infty$, *and* $0 < \max|\alpha_i/\alpha_{i+1}| < \liminf(\beta(n+1)/\beta(n)) \leqslant 1$ *where the maximum is taken over those i such that* $|\alpha_i| \neq |\alpha_{i+1}|$. *Let* D_n *be matrices with* $\|D_n\| = O(\beta(n))$. *The matrix recurrence*

$$(\mathrm{diag}(\alpha_1, ..., \alpha_k) + D_n)\, x_n = x_{n+1} \qquad (n \in \mathbb{N}) \tag{1.10}$$

has solutions $\{x_n^{(i)}\}$ *with*

$$x_n^{(i)} = \alpha_i^n e_i \left(1 + O\left(\sum_{h=n}^{\infty} \beta(h)\right)\right)$$

for $i = 1, ..., k$.

For the proof we need the following fact:

LEMMA 1.8. *Let* $\zeta \in \mathbb{R}$ *and* $\beta\colon \mathbb{N} \to \mathbb{R}_{>0}$ *a function such that* $\lim_{n\to\infty} \beta(n) = 0$ *and* $0 < \zeta < \liminf(\beta(n+1)/\beta(n)) \leqslant 1$. *Then*

$$\sum_{h=0}^{n-1} \beta(h)\, \zeta^{n-h} = O(\beta(n)).$$

Proof. Set $A = \max_n \beta(n)$ and let N be so large that $\beta(n+1)/\beta(n) > \eta$ for $n \geqslant N$, and some number $\zeta < \eta < 1$. Choose $n \geqslant N$. Since $\zeta^n/\beta(n) \to 0$ $(n \to \infty)$,

$$\frac{1}{\beta(n)} \sum_{h=N}^{n-1} \beta(h) \cdot \zeta^{n-h} + \frac{1}{\beta(n)} \sum_{h=0}^{N-1} \beta(h) \cdot \zeta^{n-h}$$
$$\leqslant \sum_{h=N}^{n-1} \left(\frac{\zeta}{\eta}\right)^{n-h} + \frac{A\zeta^n}{\beta(n)} \sum_{h=0}^{N-1} \zeta^{-h} = O(1). \qquad \text{Q.E.D.}$$

Proof of Proposition 1.7. By Theorem 1.4 there exist matrices G_n, converging to the identity matrix, such that

$$G_{n+1}^{-1}(\operatorname{diag}(\alpha_1, ..., \alpha_k) + D_n)\, G_n = \operatorname{diag}(\alpha_1, ..., \alpha_k)$$

and

$$\|G_n - I\| \leqslant C \cdot \max_{ij} T_{ij}(n) := C \cdot \max_{i,\,j} \left|\frac{\alpha_i}{\alpha_j}\right|^n \cdot \sum_{(n)} \beta(h) \left|\frac{\alpha_j}{\alpha_i}\right|^{h+1}$$

for all n and $1 \leqslant i, j \leqslant k$. We show that $T_{ij}(n) = O(\sum_{h=n}^{\infty} \beta(h))$. Let ζ be such that $\max(|\alpha_i|/|\alpha_{i+1}|) < \zeta < \liminf(\beta(n+1)/\beta(n)) \leqslant 1$, where the maximum is taken over those i such that $|\alpha_i| \neq |\alpha_{i+1}|$. For $j < i$, $T_{ij}(n) < \sum_{h=n}^{\infty} \beta(h)\, \zeta^{h-n+1}$. If $j > i$ and $\sum_{h=0}^{\infty} \beta(h)\, |\alpha_j/\alpha_i|^h$ converges, then $T_{ij}(n) = O(\zeta^n) = O(\beta(n))$. If the sum does not converge, then $T_{ij}(n) < \sum_{h=0}^{n-1} \beta(h)\, \zeta^{n-h-1} = O(\beta(n))$, by Lemma 1.8. If $i = j$, then $T_{ij}(n) = O(\sum_{h=n}^{\infty} \beta(h))$. Hence we see that

$$\|G_n - I\| = O\left(\sum_{i,\,j} T_{ij}(n)\right) = O\left(\sum_{h=n}^{\infty} \beta(h)\right).$$

Hence the matrix recurrence

$$(\operatorname{diag}(\alpha_1, ..., \alpha_k) + D_n)\, x_n = x_{n+1}$$

has solutions

$$x_n^{(i)} = \alpha_i^n G_n e_i = \alpha_i^n e_i \left(1 + O\left(\sum_{h=n}^{\infty} \beta(h)\right)\right)$$

for $i = 1, ..., k$. Q.E.D

As in Corollary 1.6 we can apply the result, which is a result for matrix recurrences, to linear recurrences (1.2) by way of the associated matrix recurrence. The fact that we can require that $\liminf(\beta(n+1)/\beta(n)) > \max(|\alpha_i|/|\alpha_{i+1}|)$ instead of $\beta(n+1)/\beta(n) \to 1$ as $n \to \infty$ was first seen by

Coffmann [1] who showed that the result of Gelfond–Kubenskaya (see the remark after Theorem 1.2) holds under this weakened condition. Notice that Proposition 1.7 in fact combines the results of [2, 3].

2. THE PROOF OF THEOREM 1.4

Theorem 1.4 tells us to what degree the entries of a sequence of diagonal matrices $\{M_n\}$ may be perturbed in order that the solutions of the matrix recurrence (1.1) $M_n x_n = x_{n+1}$ and the solutions of the "perturbed matrix recurrence" $(M_n + D_n)\, y_n = y_{n+1}$ are asymptotically equal. Namely, if a sequence $\{G_n\}$ converging to the identity matrix can be found such that $G_{n+1}^{-1}(M_n + D_n)\, G_n = M_n$ for $n \geqslant 0$, then, for $\{x_n\}$ a solution of the unperturbed recurrence (1.1), $\{y_n\} = \{G_n x_n\}$ is a solution of the perturbed matrix recurrence. Then, by $\|G_n - I\| = o(1)$, we have $y_n = x_n(1 + o(1))$.

We proved a somewhat less general version of Theorem 1.4 in [6, Lemma 4.1], but on the one hand we need the more general version in Section 7 and, on the other hand, this gives us the opportunity to repair a small flaw in the proof of the original version. We begin by stating a lemma that will be used in the proof (and which plays about the same role as does Lemma 4.2 of [6]).

LEMMA 2.1. *Let λ_n, b_n be complex numbers such that $\prod_{n=m}^{p} |\lambda_n|$ is bounded either from above or from below for all m and p and $\sum_{n=0}^{\infty} |b_n|$ converges. Then for all solutions $\{y_n\}$ of the recurrence*

$$y_{n+1} = \lambda_n y_n + b_n \qquad (n \in \mathbb{N}) \tag{2.1}$$

the estimate

$$|y_n| \leqslant \prod_{q=n_0}^{n-1} |\lambda_q| \cdot \left\{ |y_{n_0}| + \sum_{l=n_0}^{n-1} |b_l| \left(\prod_{h=n_0}^{l} |\lambda_h|^{-1} \right) \right\} \tag{2.2}$$

holds. Moreover, (2.1) *has a solution $\{w_n\}$ such that $\lim_{n \to \infty} w_n = 0$ and*

$$|w_n| \leqslant \prod_{q=n_0}^{n-1} |\lambda_q| \cdot \sum_{(n)} |b_l| \left(\prod_{h=n_0}^{l} |\lambda_h|^{-1} \right) \qquad (n \geqslant n_0), \tag{2.3}$$

where $\sum_{(n)} = \sum_{l=n_0}^{n-1}$ if the sum $\sum_{l=n_0}^{\infty}$ diverges and $\sum_{(n)} = \sum_{l=n}^{\infty}$ if the sum converges.

Proof. As can easily be checked, the solutions of the recurrence (2.1) are

$$y_n = \lambda_{n-1} \cdot \cdots \cdot \lambda_{n_0} \left(y_{n_0} + \sum_{l=n_0}^{n-1} b_l (\lambda_l \cdot \cdots \cdot \lambda_{n_0})^{-1} \right). \tag{2.4}$$

Inequality (2.2) now follows immediately. If the sum on the right-hand side converges, we take $w_{n_0} = -\sum_{l=n_0}^{\infty} b_l (\lambda_l \cdot \cdots \cdot \lambda_{n_0})^{-1}$, whence

$$w_n = -\lambda_{n-1} \cdot \cdots \cdot \lambda_{n_0} \sum_{l=n}^{\infty} b_l (\lambda_l \cdot \cdots \cdot \lambda_{n_0})^{-1}, \tag{2.5}$$

which tends to zero, if $\lambda_{n-1} \cdot \cdots \cdot \lambda_{n_0}$ is bounded from above. If $\lambda_{n-1} \cdot \cdots \cdot \lambda_{n_0}$ is not bounded from above, it is bounded from below, and in that case w_n also tends to zero, by

$$w_n = -\sum_{l=n}^{\infty} b_l (\lambda_l \cdot \cdots \cdot \lambda_n)^{-1}.$$

The estimate (2.3) follows immediately. If the sum on the right-hand side diverges, we must have $\prod_{h=n_0}^{\infty} \lambda_h = 0$ (otherwise the products $\prod_{h=n_0}^{l} |\lambda_h|^{-1}$ would be bounded from above). Choose $w_{n_0} = 0$. Again, (2.3) is immediate, and

$$|w_n| \leqslant \prod_{q=n_0}^{n-1} |\lambda_q| \cdot \sum_{l=n_0}^{n_1-1} |b_l| \cdot \left(\prod_{h=n_0}^{l} |\lambda_h|^{-1} \right) + \sum_{l=n_1}^{n-1} |b_l| \cdot \prod_{h=l+1}^{n-1} |\lambda_h|,$$

where n_1 is chosen such that $\sum_{h=n_1}^{\infty} |b_h| < \varepsilon$ for some fixed $\varepsilon > 0$. The first term on the right-hand side converges to zero as n goes to infinity, whereas the second term is bounded by a constant times ε. Q.E.D

Proof of Theorem 1.4. (1) We first prove the theorem for $L = 0$. Set

$$d(n) = \max_{1 \leqslant j \leqslant k} \frac{\|D_n\|}{|a_j(n)|}$$

and

$$\Lambda_n = \max_{i,\, j} \prod_{q=0}^{n-1} \left| \frac{a_i(q)}{a_j(q)} \right| \cdot \sum_{(n)} d(l) \prod_{h=0}^{l} \left| \frac{a_j(h)}{a_i(h)} \right|, \tag{2.6}$$

where the maximum is taken over all pairs $1 \leqslant i, j \leqslant k$. Then $\sum_{n=0}^{\infty} d(n)$ converges and Λ_n tends to zero as $n \to \infty$. Indeed, for all i, j the expression in (2.6) on the right of $\max_{i,j}$ is a solution of the recurrence

$$y_{n+1} = \left| \frac{a_i(n)}{a_j(n)} \right| y_n + d(n) \qquad (n \in \mathbb{N})$$

that tends to zero as $n \to \infty$, by Lemma 2.1. Further, let $c > 1$ be such that

$$\frac{1}{c} \|M\| \leqslant \max_{1 \leqslant i, j \leqslant k} |A_{ij}| \leqslant c \, \|M\| \tag{2.7}$$

for any matrix $M \in K^{k,k}$. That such a c exists is guaranteed by the equivalence of matrix norms in $K^{k,k}$. Let $N \in \mathbb{N}$ be so large that $\Lambda_n < 1/20c^2$ for $n \geqslant N$. We define sequences of $k \times k$-matrices $\{G_n^{(j)}\}$ as

$$G_n^{(0)} = I_k \qquad (n \geqslant 0)$$

and, for $j \geqslant 0$, $n \geqslant N$,

$$\begin{aligned} G_n^{(j+1)} - I_k = {} & G_n^{(j)} A_{n-1} \cdot \cdots \cdot A_N \\ & \times \Bigg[G_N^{(j+1)} - I_k + \sum_{l=N}^{n-1} (A_l \cdot \cdots \cdot A_N)^{-1} \\ & \times (G_{l+1}^{(j)})^{-1} D_l A_l^{-1} (A_l \cdot \cdots \cdot A_N) \Bigg] (A_{n-1} \cdot \cdots \cdot A_N)^{-1}, \end{aligned} \tag{2.8}$$

where

$$(G_N^{(j+1)} - I_k)_{pq} = 0$$

if the sum

$$\sum_{l=0}^{\infty} d(l) \left(\prod_{h=0}^{l} \left| \frac{a_q(h)}{a_p(h)} \right| \right) \tag{2.9}$$

diverges and

$$\begin{aligned} & (G_N^{(j+1)} - I_k)_{pq} \\ & \quad = -\left(\sum_{l=N}^{\infty} (A_l \cdot \cdots \cdot A_n)^{-1} (G_{l+1}^{(j)})^{-1} D_l A_l^{-1} (A_l \cdot \cdots \cdot A_N) \right)_{pq} \end{aligned}$$

if the sum (2.9) converges.

We first show that $\{G_n^{(j)}\}$ converges to I_k as $n \to \infty$ and, in addition, $\|G_n^{(j)} - I_k\| < 1/5$ for all $j \geqslant 0$ and $n \geqslant N$. For $j = 0$ this is trivial. Suppose it is true for $G_n^{(j)}$. In particular, it follows that $\|G_n^{(j)}\| < 2$, $\|(G_n^{(j)})^{-1}\| < 2$. By (2.8),

$$|((G_n^{(j)})^{-1}(G_n^{(j+1)} - I_k))_{pq}|$$
$$\leqslant \prod_{h=N}^{n-1} \left|\frac{a_p(h)}{a_q(h)}\right| \cdot \sum_n |((G_{l+1}^{(j)})^{-1} D_l A_l^{-1})_{pq}| \cdot \prod_{h=N}^{l} \left|\frac{a_q(h)}{a_p(h)}\right|,$$

where $\sum_n$ stands for $\sum_{l=N}^{n-1}$ if the sum in (2.6) diverges, and for $\sum_{l=n}^{\infty}$ if the sum in (2.6) converges. Hence, by (2.7),

$$\|(G_n^{(j)})^{-1}(G_n^{(j+1)} - I_k)\| \leqslant 2c^2 \Lambda_n \tag{2.10}$$

and

$$\|G_n^{(j+1)} - I_k\| \leqslant 4c^2 \Lambda_n < \tfrac{1}{5} \qquad (n \geqslant N). \tag{2.11}$$

It follows that $G_n^{(j+1)} \to I_k$ as $n \to \infty$.

We now show that for $n \geqslant N$ the sequences $\{G_n^{(j)}\}_j$ converge to a limit. Set $M_j = \sup_{n \geqslant N} \|G_n^{(j)} - G_n^{(j-1)}\|$ $(j \geqslant 1)$. Note that $M_1 \leqslant 1/5$. Further,

$$\|(G_n^{(j)})^{-1} - (G_n^{(j-1)})^{-1}\| = \|(G_n^{(j)})^{-1}(G_n^{(j-1)} - G_n^{(j)})(G_n^{(j-1)})^{-1}\| \leqslant 4M_j.$$

Then

$$M_{j+1} \leqslant M_j \cdot \|(G_n^{(j)})^{-1}(G_n^{(j+1)} - I_k)\|$$
$$+ \Bigg\| G_n^{(j-1)} A_{n-1} \cdot \cdots \cdot A_N \Bigg(G_N^{(j)} - G_N^{(j+1)} + \sum_{l=N}^{n-1} (A_l \cdot \cdots \cdot A_N)^{-1}$$
$$\times ((G_{l+1}^{(j-1)})^{-1} - (G_{l+1}^{(j)})^{-1}) D_l A_l^{-1} (A_l \cdot \cdots \cdot A_N) \Bigg)$$
$$\times (A_{n-1} \cdot \cdots \cdot A_N)^{-1} \Bigg\|$$

whence, by the definition of the numbers $G_N^{(i)}$ and by (2.7) and (2.10),

$$M_{j+1} \leqslant (2c^2 \max_{n \geqslant N} \Lambda_n + 8c^2 \max_{n \geqslant N} \Lambda_n)\, M_j \leqslant \tfrac{1}{2} M_j \qquad (j \geqslant 1).$$

If we now set, for $n \geqslant N$,

$$G_n = G_n^{(0)} + \sum_{j=0}^{\infty} (G_n^{(j+1)} - G_n^{(j)}),$$

then $\|G_n\| = \|G_n^{(0)}\| + \sum_{j=1}^{\infty} M_j < \infty$ and

$$\|G_n - G_n^{(r)}\| \leqslant \sum_{j=r+1}^{\infty} M_j \leqslant 2^{1-r}/5$$

so that $\{G_n^{(r)}\}_r$ converges to G_n for $n \geqslant N$ as $r \to \infty$. In addition, the estimate (2.11) holds for all $G_n^{(j+1)}$, hence for G_n. This yields (1.9). It remains to be shown that $A_n + D_n = G_{n+1} A_n G_n^{-1}$. If we take limits in (2.8), letting $j \to \infty$, we find

$$G_n - I_k = G_n A_{n-1} \cdot \cdots \cdot A_N \left(G_N - I_k + \sum_{l=N}^{n-1} (A_l \cdot \cdots \cdot A_N)^{-1} \right.$$
$$\left. \times G_{l+1}^{-1} D_l A_l^{-1} (A_l \cdot \cdots \cdot A_N) \right) (A_{n-1} \cdot \cdots \cdot A_N)^{-1} \quad (2.12)$$

whence

$$G_{n+1} - I_k = G_{n+1} A_n G_n^{-1} (G_n - I_k) A_n^{-1} + D_n A_n^{-1}$$

so that

$$G_{n+1} A_n G_n^{-1} A_n^{-1} = I_k + D_n A_n^{-1}$$

and from this it is easy to see that $A_n + D_n = G_{n+1} A_n G_n^{-1}$. This identity can, moreover, be used to define G_n recursively for $n = N-1, ..., n = 0$.

(2) We now prove the general case. Before we proceed, it is useful to introduce the following convention for matrix norms of submatrices of matrices in $K^{k,k}$: given some submatrix B that is composed of a selected subset of the rows and columns of a given matrix $A \in K^{k,k}$, we extend B to a matrix $C \in K^{k,k}$ by letting the entries in the rows or columns that do not occur in B be zero. Then we let $\|B\| = \|C\|$. It may be assumed that $\sum_{l=0}^{\infty} (\|D_l\|/|a_j(l)|)$ converges for $j > L$ and diverges for $j \leqslant L$. We set

$$d(n) = \max_{L+1 \leqslant j \leqslant k} \frac{\|D_n\|}{|a_j(n)|}$$

and we define Λ_n by (2.6), except that the maximum is taken over all pairs i, j with $L+1 \leqslant i, j \leqslant k$, and similarly we define Λ'_n by (2.6), where the maximum is taken over all pairs i, j with $1 \leqslant i \leqslant L$, $L+1 \leqslant j \leqslant k$. Note that in particular it follows that $d(n) \leqslant \sum_{(n)} d(l) \leqslant \Lambda_n$ for $n \geqslant N$. As in part (1) of the proof, $\sum_{l=0}^{\infty} d(l)$ converges and both Λ_n and Λ'_n tend to zero as

$n \to \infty$. We let $N \in \mathbb{N}$ be so large that $\Lambda_n < 1/40c^2$ and $\Lambda'_n < 1/4c^2$ for $n \geqslant N$, where c is as in (2.7). Put

$$A_n + D_n = \begin{pmatrix} P_n & Q_n \\ R_n & S_n \end{pmatrix} \qquad (n \in \mathbb{N})$$

where $P_n \in K^{L,L}$, $S_n \in K^{k-L,k-L}$. We show that the equation

$$X_{n+1} = (P_n X_n + Q_n)(S_n + R_n X_n)^{-1} \qquad (X_n \in K^{L,k-L}, n \geqslant N) \tag{2.13}$$

has a solution $\{X_n\}$ that converges to zero. Set

$$P'_n = \operatorname{diag}(a_1(n), ..., a_L(n)), \qquad S'_n = \operatorname{diag}(a_{L+1}(n), ..., a_k(n)),$$

and

$$Q'_n = (P_n - P'_n)\, X_n + Q_n \qquad (n \geqslant N).$$

We define sequences $\{X_n^{(j)}\}$, $\{Q_n^{(j)}\}$, $\{S_n^{(j)}\}$ for $j \geqslant 0$, $n \geqslant N$ by $X_n^{(0)} = 0$, $X_N^{(j)} = 0$ for all j and

$$Q_n^{(j)} = (P_n - P'_n)\, X_n^{(j)} + Q_n, \qquad S_n^{(j)} = S_n + R_n X_n^{(j)},$$
$$X_{n+1}^{(j+1)} = (P'_n X_n^{(j+1)} + Q_n^{(j)})(S_n^{(j)})^{-1}.$$

We first show that $\{X_n^{(j)}\}$ converges to zero as $n \to \infty$ and that $\|X_n^{(j)}\| \leqslant 1$ for all j and $n \geqslant N$, and further, that $S_n^{(j)}$ is invertible, so that the sequences are well-defined. For $j = 0$ this is trivial and we suppose that it is true for $\{X_n^{(j)}\}$. Then

$$\|Q_n^{(j)}\| \leqslant 2\, \|D_n\|,$$

$$\|S_n^{(j)} - S'_n\| \leqslant 2\, \|D_n\| \leqslant \frac{1}{20c^2} \min_{L+1 \leqslant i \leqslant k} |a_i(n)| < \|(S'_n)^{-1}\|^{-1},$$

so that $S_n^{(j)}$ is indeed invertible for all $n \geqslant N$, and

$$\|(S_n^{(j)})^{-1}\| \leqslant \frac{\|(S'_n)^{-1}\|}{1 - 2\, \|(S'_n)^{-1}\| \cdot \|D_n\|}. \tag{2.14}$$

We may apply part (1) of the proof to $S_n^{(j)}$ and obtain some sequence $\{H_n^{(j)}\}$ in $K^{k-L,k-L}$ such that $\|H_n^{(j)} - I_{L-k}\| \leqslant 8c^2\Lambda_n \leqslant 1/5$ and $S_n^{(j)} = H_{n+1} S'_n (H_n^{(j)})^{-1}$. An explicit expression for $X_n^{(j+1)}$ is given by

$$X_n^{(j+1)} = \sum_{l=N}^{n-1} P'_{n-1} \cdot \cdots \cdot P'_{l+1} Q_l^{(j)} H_l^{(j)} (S'_l)^{-1} (S'_{n-1} \cdot \cdots \cdot S'_{l+1})^{-1} (H_n^{(j)})^{-1},$$

whence

$$|(X_n^{(j+1)}H_n^{(j)})_{pq}| \leqslant \sum_{l=N}^{n-1} |(Q_l^{(j)}H_l^{(j)}(S_l')^{-1})_{pq}| \prod_{h=l+1}^{n-1} \left|\frac{a_p(h)}{a_{q+L}(h)}\right|$$

for $1 \leqslant p \leqslant L$, $1 \leqslant q \leqslant k-L$ and, by $\|H_n^{(j)}\| \leqslant 6/5$, $\|(H_n^{(j)})^{-1}\| \leqslant 6/5$,

$$\|X_n^{(j+1)}\| \leqslant 4c^2 \Lambda_n' < 1. \tag{2.15}$$

We show that, for n fixed, the sequences $\{X_n^{(j)}\}_j$ converge to limits X_n. For $n = N$ this is obvious and $X_N = 0$. Suppose it is true for $N \leqslant n \leqslant m$. Because

$$X_{m+1}^{(j+1)} = (P_m' X_m^{(j+1)} + (P_m - P_m')\, X_m^{(j)} + Q_m)(S_m + R_m X_m^{(j)})^{-1} \tag{2.16}$$

for $j \geqslant 0$ and by (2.14), we can take limits in (2.16) and let $j \to \infty$. Then the right-hand side converges to the right-hand side of (2.13) and thus $X_{n+1}^{(j)}$ converges to some limit X_{n+1} such that (2.13) holds. In addition, by (2.15),

$$\|X_{n+1}\| \leqslant 4c^2 \Lambda_{n+1}' < 1.$$

In particular, $X_n \to 0$ as $n \to \infty$. (Note that for all $1 \leqslant i \leqslant L$, $L+1 \leqslant j \leqslant k$ the sums that occur in the definition of Λ_n' do not converge, because otherwise $\sum_{n=0}^{\infty} (\|D_n\|/|a_i(n)|)$ would converge for at least one $i \leqslant N$, which is in conflict with the assumption. Hence $\sum_{(n)}$ stands for $\sum_0^{n-1}$ for these pairs i, j in (1.9).)

Setting

$$H_n^{(1)} = \begin{pmatrix} I_L & X_n \\ 0 & I_{k-L} \end{pmatrix} \qquad (n \geqslant N)$$

we have that $\|H_n^{(1)} - I\| = \|X_n\|$ and

$$(H_{n+1}^{(1)})^{-1}\,(A_n + D_n)\,H_n^{(1)} = \begin{pmatrix} P_n - X_{n+1}R_n & 0 \\ R_n & R_n X_n + S_n \end{pmatrix}. \tag{2.17}$$

We now go on to show that the equation

$$Y_{n+1}(P_n - X_{n+1}R_n) = (R_n X_n + S_n)\, Y_n + R_n \qquad (Y_n \in K^{L-k,\,L},\ n \geqslant N)$$

has a solution $\{Y_n\}$ that converges to zero. As before, we use an iteration method. Let P_n', S_n' be as above and set

$$\Lambda_n'' = \max_{i,\,j} \prod_{q=0}^{n-1} \left|\frac{a_i(q)}{a_j(q)}\right| \cdot \sum_{(n)} d(l) \prod_{h=0}^{l-1} \left|\frac{a_j(h)}{a_i(h)}\right|,$$

where the maximum is taken over all pairs i, j with $L+1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant L$. Again, Λ''_n tends to zero as $n \to \infty$, by Lemma 2.1. Note that the sum converges and $\Lambda''_n \leqslant c_1 \Lambda_n$ for some constant $c_1 > 0$. Further we let $N' \geqslant N$ be so large that $\Lambda''_n \leqslant 1/5c^2$ for $n \geqslant N'$. For simplicity, we write N for N'.

We define sequences $\{Y_n^{(j)}\}$, $\{R_n^{(j)}\}$ $(n \geqslant N, j \geqslant 0)$ by

$$R_n^{(j)} = R_n + (X_{n+1} R_n + P'_n - P_n)\, Y_{n+1}^{(j)} + (R_n X_n + S_n - S'_n)\, Y_n^{(j)} \tag{2.18}$$

and

$$Y_n^{(0)} = 0, \qquad Y_{n+1}^{(j+1)} P'_n = S'_n Y_n^{(j+1)} + R_n^{(j)} \qquad (n \geqslant N, j \geqslant 0).$$

An explicit expression for $Y_n^{(j)}$ is given by

$$Y_n^{(j)} = (S'_{n-1} \cdot \cdots \cdot S'_N) \left(Y_N^{(j)} + \sum_{l=N}^{n-1} (S'_{l-1} \cdot \cdots \cdot S'_N)^{-1} (S'_l)^{-1} R_l^{(j-1)} \right.$$

$$\left. \times (P'_{l-1} \cdot \cdots \cdot P'_N) \right) (P'_{n-1} \cdot \cdots \cdot P'_N)^{-1}.$$

If $\|R_n^{(j-1)}\| \leqslant 5\, \|D_n\|$ for $n \geqslant N$, then

$$\left\| \sum_{l=N}^{n-1} (S'_{l-1} \cdot \cdots \cdot S'_N)^{-1} (S'_l)^{-1} R_l^{(j-1)} (P'_{l-1} \cdot \cdots \cdot P'_N) \right\|$$

$$\leqslant 5c^2 \max_{1 \leqslant i \leqslant L,\, L+1 \leqslant j \leqslant k} \sum_{l=N}^{n-1} d(l) \prod_{h=N}^{l-1} \left| \frac{a_i(h)}{a_j(h)} \right|$$

and the sum on the right-hand side converges, because the quotients $\prod_{h=N}^{l-1} (|a_i(h)/a_j(h)|)$ are bounded. If we choose

$$Y_N^{(j)} = -\sum_{l=N}^{\infty} (S'_{l-1} \cdot \cdots \cdot S'_N)^{-1} (S'_l)^{-1} R_l^{(j-1)} (P'_{l-1} \cdot \cdots \cdot P'_N),$$

then

$$Y_n^{(j)} = -\sum_{l=n}^{\infty} (S'_{l-1} \cdot \cdots \cdot S'_n)^{-1} (S'_l)^{-1} R_l^{(j-1)} (P'_{l-1} \cdot \cdots \cdot P'_n), \tag{2.19}$$

whence

$$\|Y_n^{(j)}\| \leqslant 5c^2 \Lambda''_n \leqslant 1.$$

In particular, $Y_n^{(j)} \to 0$ as $n \to \infty$. Further, by (2.18) we also have $\|R_n^{(j)}\| \leqslant 5\, \|D_n\|$ for $n \geqslant N$.

We show that the sequences $\{Y_n^{(j)}\}_j$ converge to limits Y_n as $j \to \infty$. Let $m_j = \max_{n \geqslant N} \| Y_n^{(j)} - Y_n^{(j-1)} \|$. We have $m_1 \leqslant 5c^2 \Lambda''_N \leqslant 1$. By (2.18),

$$\|(S'_l)^{-1} (R_l^{(j-1)} - R_l^{(j-2)})\| \leqslant 4m_{j-1} d(l).$$

Using expression (2.19) we find that, for $j \geqslant 2$, $1 \leqslant p \leqslant k-L$, $1 \leqslant q \leqslant L$,

$$|(Y_n^{(j)} - Y_n^{(j-1)})_{pq}| \leqslant \sum_{l=n}^{\infty} |((S'_l)^{-1} (R_l^{(j-1)} - R_l^{(j-2)})_{pq})| \prod_{h=n}^{l-1} \frac{|a_q(h)|}{|a_{p+L}(h)|},$$

so that

$$\| Y_n^{(j)} - Y_n^{(j-1)} \| \leqslant 4c^2 m_{j-1} \Lambda''_n,$$

whence

$$m_j \leqslant 4c^2 m_{j-1} \max_{n \geqslant N} \Lambda''_n \leqslant \tfrac{4}{5} m_{j-1}.$$

If we define $Y_n = \sum_{j=0}^{\infty} (Y_n^{(j+1)} - Y_n^{(j)})$ for $n \geqslant N$, which is well-defined in view of (2.18), then clearly $Y_n^{(j)} \to Y_n$ as $j \to \infty$, and

$$\| Y_n \| \leqslant 5c^2 \Lambda''_n \leqslant 1 \qquad (n \geqslant N).$$

Setting

$$H_n^{(2)} = \begin{pmatrix} I_L & 0 \\ Y_n & I_{k-L} \end{pmatrix} \qquad (n \geqslant N),$$

we have that $\|H_n^{(2)} - I\| = \| Y_n \|$ and

$$(H_{n+1}^{(1)} H_{n+1}^{(2)})^{-1} (A_n + D_n) H_n^{(1)} H_n^{(2)} = \begin{pmatrix} P_n - X_{n+1} R_n & 0 \\ 0 & R_n X_n + S_n \end{pmatrix}.$$

Finally, we note that $R_n X_n + S_n$ is invertible and $\sum_n (\| R_n X_n + S_n - S'_n \| / |a_j(n)|) < \infty$ for $j > L$ by (2.14), hence we may apply part (1) of the proof to the matrix block $R_n X_n + S_n$, thus finding matrices $G'_n \in K^{k-L, k-L}$ $(n \geqslant N)$ such that

$$(G'_{n+1})^{-1} (R_n X_n + S_n) G'_n = \operatorname{diag}(a_{L+1}(n), ..., a_k(n)) = S'_n$$

and

$$\|G'_n - I\| \leqslant c_2 \Lambda_n \qquad (n \geqslant N)$$

for some constant $c_2 > 0$. Finally, we set

$$G_n = H_n^{(1)} H_n^{(2)} \begin{pmatrix} I_L & 0 \\ 0 & G'_n \end{pmatrix}$$

and $Z_n = -X_{n+1}$. The estimate for Z_n follows from $\|X_n\| \leqslant 4c^2 \Lambda'_n$.

3. MÖBIUS-TRANSFORMATIONS: GENERAL PROPERTIES

A way to study matrix recurrences $M_n x_n = x_{n+1}$ of order two is to consider the matrix M_n, instead of representing a linear map from $\mathbb{C}^2$ to $\mathbb{C}^2$, as representing a map from the one-dimensional complex projective space $\mathbb{P}^1(\mathbb{C})$ to itself. In fact, if $M_n = \left(\begin{smallmatrix} a_n & b_n \\ c_n & d_n \end{smallmatrix}\right)$, we let $F_n(z) = (a_n z + b_n)/(c_n z + d_n)$ for $n \in \mathbb{N}$. Instead of studying the behaviour of solutions of the matrix recurrence, we study the orbits of points in $\mathbb{P}^1(\mathbb{C})$ under the action of the sequence of Möbius-transformations (or fractional linear maps, as they are also called) $\{F_n\}$. The advantage of this point of view is that we can use the topology of $\mathbb{P}^1(\mathbb{C})$. Note that F_n is not a constant map because $\det M_n \neq 0$. First we recall a few classical, but important properties of Möbius-transformations. All of them can no doubt be found in the literature, but it is useful to put them together for easy reference.

We take the usual topology on $\mathbb{P}^1(\mathbb{C})$, i.e., we take the usual topology on $\mathbb{C}$, and let $\mathbb{P}^1(\mathbb{C})$ be the one-point compactification. It is well known that $\mathbb{P}^1(\mathbb{C})$ can be identified with a sphere S^2 in $\mathbb{R}^3$, e.g., by stereographic projection. This enables us to define a distance on $\mathbb{P}^1(\mathbb{C})$, which is invariant under maps in $SU(2, \mathbb{C})$, namely, let $d(z, w) = |z - w|/(1 + |z|^2)^{1/2} (1 + |w|^2)^{1/2}$ for $z, w \in \mathbb{C}$ and $d(z, \infty) = 1/(1 + |z|^2)^{1/2}$ for $z \in \mathbb{C}$. $d(z, w)$ comes from the usual distance on the sphere in $\mathbb{R}^3$, by stereographic projection. The following lemma collects some essential properties and identities that we shall need in the sequel.

LEMMA 3.1. (1) *A Möbius-transformation is a homeomorphism of* $\mathbb{P}^1(\mathbb{C})$.

(2) *Möbius-transformations leave the harmonic double ratio* $(z_1, z_2; z_3, z_4) = (z_1 - z_3)/(z_1 - z_4) : (z_2 - z_3)/(z_2 - z_4)$ *invariant.*

(3) *A Möbius-transformation which is not the identical map* $F(z) = z$ *has either one or two fixpoints in* $\mathbb{P}^1(\mathbb{C})$.

(4) *For F a Möbius–transformation, $w, z \neq \infty$, we have*

$$(F(z) - F(w))^2 = F'(z)\, F'(w)(z - w)^2.$$

(5) *If F has two fixpoints ζ and η, both $\neq \infty$, then $F'(\zeta)\, F'(\eta) = 1$. If F has one fixpoint ζ, then $F'(\zeta) = 1$.*

(6) *If F has two fixpoints ζ, η and $\zeta \neq \infty$, then*

$$\frac{F(z)-\zeta}{F(z)-\eta}=F'(\zeta)\cdot\frac{z-\zeta}{z-\eta}. \tag{3.1}$$

If F has only one fixpoint $\zeta \neq \infty$, then

$$\frac{1}{F(z)-\zeta}=\frac{1}{z-\zeta}+c \tag{3.2}$$

for some $c \in \mathbb{C}$.

Proof. Parts (1)–(3) are classical, and can be found, e.g., in [12, 16]. I have not been able to find a reference for (4)–(6) but the proofs are very simple.

(4) This follows from (2) by taking $z_1=z$, $z_2=w+h$, $z_3=w$, $z_4=z+h$, and sending h to 0.

(5) This follows from (4) by taking $z=\zeta$, $w=\eta$. For F parabolic, see (6).

(6) From (4) and (5) it follows that

$$\left(\frac{F(z)-\zeta}{F(z)-\eta}\right)^2=F'(\zeta)^2\cdot\left(\frac{z-\zeta}{z-\eta}\right)^2$$

from which (6) follows up to a plus or minus sign. That the sign must be plus follows if we realize that, by continuity, the sign must be independent of z. Taking $z=\zeta+h$ and sending $h\to 0$ shows that the sign is indeed positive.

Now suppose that F is parabolic with fixpoint ζ. Let $G(z)=1/(z-\zeta)$. Then GFG^{-1} is parabolic with fixpoint ∞. It is easy to see that a parabolic map with fixpoint ∞ is of the form $z\to z+c$ for some $c\in\mathbb{C}$. Formula (3.2) now follows immediately. Furthermore, formula (3.2) shows us that $F'(\zeta)=1$ if $\zeta\neq\infty$. Q.E.D

We recall Klein's classification of Möbius-transformations (see, e.g., [12, 16]): If F has exactly one fixpoint, it is called parabolic; if F has exactly two fixpoints ζ, η, then it is called hyperbolic if $F'(\zeta)$ is real and not 1 or -1; it is called elliptic if $|F'(\zeta)|=1$; and if $|F'(\zeta)|\neq 1$ and $F'(\zeta)$ is not real, it is called loxodromic. If $\zeta=\infty$ we can define, with abuse of notation, $F'(\zeta)=F'(\eta)^{-1}$ (or $=1$ if F is parabolic). This is consistent with the fact that if G is another Möbius-transformation, then $G(\zeta)$ is a fixpoint of GFG^{-1} if ζ is a fixpoint of F and $(GFG^{-1})'(G(\zeta))=F'(\zeta)$. As we shall see in the next sections, the numbers $F'(\zeta)$, for ζ a fixpoint of F, are a very important indicator of the way in which the solutions converge. Finally, if F is a Möbius-transformation, associated to a matrix $M\in GL(2,\mathbb{C})$ in the natural way (as indicated above), then the fixpoints of F correspond to the eigenvectors of M ($(y_1, y_2)^t$ is an eigenvector of M if and only if y_1/y_2 is a fixpoint of F (where $1/0 = \infty$)) and the numbers $F'(\zeta)$ correspond to the quotients of eigenvalues of M. The easiest way to see this is to consider GFG^{-1} with fixpoints 0 and ∞ (or only ∞, if F is parabolic) instead of F. In this case, GFG^{-1} is of the form $GFG^{-1}(z) = \lambda z$ (or $GFG^{-1}(z) = z + c$ for some complex number c if F is parabolic).

4. SEQUENCES OF MÖBIUS-TRANSFORMATIONS

We now consider sequences of Möbius-transformations $\{F_n\}$. In this section we derive some results that enable us to study the behaviour of solutions of recurrences

$$F_n(z_n) = z_{n+1} \qquad (n \in \mathbb{N}). \tag{4.1}$$

First of all, in order to get a global impression of what one can expect, we study the behaviour of solutions of recurrences $F(z_n) = z_{n+1}$, i.e., where the sequence $\{F_n\}$ is a constant sequence. We shall see that the number $F'(\zeta)$, where ζ is one of the fixpoints of F, plays a crucial role. Using formula (4.1), we see that

$$\frac{z_n - \zeta}{z_n - \eta} = F'(\zeta)^n \cdot \frac{z_0 - \zeta}{z_0 - \eta},$$

where ζ, η are the fixpoints of F. We derive immediately that if F is hyperbolic or loxodromic, and ζ is the fixpoint for which $|F'(\zeta)| < 1$, then all solutions $\{z_n\}$ but one converge to ζ. The one remaining solution is the constant solution $\{\eta\}$. On the other hand, if F is elliptic, then except for the two constant solutions $\{\zeta\}$ and $\{\eta\}$, none of the solutions converge to the fixpoints, but remain on fixed circles $\{z \in \mathbb{C}: |(z-\zeta)/(z-\eta)| = c\}$ $(c \in \mathbb{R})$. If F is parabolic, then formula (2.2) shows that

$$\frac{1}{z_n - \zeta} = \frac{1}{z_0 - \zeta} + nc,$$

so that all solutions converge to the fixpoint ζ.

DEFINITION. A fixpoint ζ of a Möbius-transformation F is called *hyperbolic*, if F is hyperbolic or loxodromic. It is called *elliptic* if F is elliptic, and it is called *parabolic*, if F is parabolic. If ζ is a hyperbolic fixpoint, it is called *attracting* (resp. *repelling*) if $|F'(\zeta)| < 1$ (resp. $|F'(\zeta)| > 1$).

DEFINITION. A point $\zeta \in \mathbb{P}^1(\mathbb{C})$ is called an *attracting* point of the recurrence (4.1) if there is a neighbourhood U of ζ and some number N such that any solution $\{z_n\}$ that enters U for some $n \geqslant N$ (i.e., $z_n \in U$) converges to ζ.

The following lemma gives four elementary, but important, facts about convergence of the solutions of a recurrence $z_{n+1} = F_n(z_n)$.

LEMMA 4.1. *Consider the recurrence*

$$z_{n+1} = F_n(z_n) \qquad (n \in \mathbb{N}) \tag{4.1}$$

for $\{F_n\}$ *a sequence of Möbius-transformations.*

(1) *If two solutions* $\{z_n^{(1)}\}$ *and* $\{z_n^{(2)}\}$ *of the recurrence converge to some limit point* $\xi \in \mathbb{P}(\mathbb{C})$, *then for any two solutions* $\{w_n^{(1)}\}$ *and* $\{w_n^{(2)}\}$ *of the same recurrence and for any* $\varepsilon > 0$ *there is a number* $N \in \mathbb{N}$ *such that for* $n \geqslant N$ *either* $d(w_n^{(1)}, \xi) < \varepsilon$ *or* $d(w_n^{(2)}, \xi) < \varepsilon$ *(or both).*

(2) *If the recurrence has three solutions, two of which converge to some limit point* ζ, *whereas the other one converges to some other limit point* η, *then all solutions except one converge to* ζ.

(3) *If the recurrence has three solutions, converging to three distinct limits, then all solutions must converge, and any* $w \in \mathbb{P}^1(\mathbb{C})$ *is the limit point of exactly one solution.*

(4) *If the sequence* $\{F_n\}$ *converges to some limit* F, *and the recurrence has some solution* $\{z_n\}$ *converging to a limit point* ξ, *the* ξ *must be a fixpoint of* F.

Proof. (1) We may suppose that all four solutions are distinct. Suppose that $d(w_{n_i}^{(j)}, \xi) > \varepsilon$ for $j = 1, 2$ and $n_i \to \infty$. Then either $|w_{n_i}^{(j)} - \xi| > a \cdot \varepsilon$ or $|1/w_{n_i}^{(j)} - 1/\xi| > a \cdot \varepsilon$ for some $a > 0$. Hence the harmonic double ratios $(z_{n_i}^{(1)}, z_{n_i}^{(2)}; w_{n_i}^{(1)}, w_{n_i}^{(2)})$ converge to 1 as $i \to \infty$, which is impossible.

(2) This is an immediate consequence of (1).

(3) This again follows simply from the invariance of the harmonic double ratio.

(4) By continuity, $F_n(z_n)$ must converge to $F(\xi)$. Q.E.D

The following corollary follows directly from Lemma 4.1(1).

COROLLARY 4.2. *If the recurrence* (4.1) *has an attracting point* $\zeta \in \mathbb{P}^1(\mathbb{C})$, *then there is at most one solution of* (4.1) *that does not converge to* ζ.

If F is not "close" to the identity map, then $d(F(z), z)$ is small only if z is close to some fixpoint of F. Hence in general, if the solutions of a recurrence

$F_n(z_n) = z_{n+1}$ converge to some limit point, either this limit point must be the limit of fixpoints of F_n, or (a subsequence of) the sequence $\{F_n\}$ converges to the identity map. In the section of examples, we shall show that it is possible that the fixpoints of the maps F_n converge, but that the solutions do not converge at all (e.g., Examples 9.3 and 9.5). or even converge to other limit points (Example 9.2). Moreover, it is possible that there are infinitely many converging solutions and infinitely many non-converging solutions. In order to have convergence of the solutions to the limits of the fixpoints, we must impose conditions on the way the fixpoints converge and on the numbers $F'_n(\zeta_n)$ as well (ζ_n a fixpoint of F_n). Situations where we can obtain neat convergence results occur if the fixpoints are, in a sense to be specified, "sufficiently hyperbolic" (see Example 6.4), and also if the fixpoints are *of bounded variation*, i.e., if $\sum_n d(\zeta_n, \zeta_{n+1})$ and $\sum_n d(\eta_n, \eta_{n+1})$ converge for the fixpoints ζ_n, η_n of F_n and if they do not converge to the same limit. In order to study the latter case, we shall first have a look at the case where the F_n have one fixpoint in common. This is the subject of the next section.

5. INHOMOGENEOUS FIRST-ORDER RECURRENCES

In the preceding section we studied the asymptotic behaviour of solutions of a recurrence given by a sequence of Möbius-transformations, where all Möbius-transformations are the same. We have seen that there are, grossly speaking, three types of behaviour (hyperbolic, elliptic., and parabolic) but in either case, there are solutions that converge to the fixpoints. The next difficult case we turn our attention to is the case where the Möbius-transformations are not the same, but have one fixpoint in common. In this case there is an explicit formula for the solutions of the recurrence, which allows us to study in detail what types of asymptotic behaviour can occur. We shall see that in this case there is a much richer variety of asymptotic behaviour, even if both fixpoints of the Möbius-transformations converge: in that case there are not always converging solutions (except for the constant solution $\{\infty\}$ of course); see Section 9 for examples where the convergence behaviour is not so neat. The most important result of this section is that if the fixpoints are of bounded variation (and not converging to equal limits), the asymptotic behaviour of the solutions depends entirely on the derivatives of the Möbius-transformations in the fixpoints, a result that will also hold if the Möbius-transformations do not have a fixpoint in common. But this will have to wait until Section 7.

If F is a Möbius-transformation with infinity as a fixpoint, then F is of the form $F(z) = az + b$ with a, b complex numbers. If F has $\eta \neq \infty$ as a fixpoint, and $G(z) = 1/(z - \eta)$, then GFG^{-1} has infinity as a fixpoint. In particular, we have $G(F(z)) = a \cdot G(z) + b$. Suppose F has another fixpoint $\zeta \neq \infty$. Then $G'(\zeta) \cdot F'(\zeta) = aG'(\zeta)$, hence $a = F'(\zeta)$ and $b = G(\zeta)(1 - F'(\zeta))$:

$$\frac{1}{F(z) - \eta} = F'(\zeta) \cdot \frac{1}{z - \eta} + \frac{(1 - F'(\zeta))}{\zeta - \eta}. \tag{5.1}$$

If $\eta = \infty$, then G is the identity, so that just

$$F(z) = F'(\zeta)\, z + (1 - F'(\zeta))\, \zeta. \tag{5.2}$$

In this section, we investigate the solutions of a recurrence $F_n(z_n) = z_{n+1}$ where the F_n have a common fixpoint η. As is clear from the preceding discussion, we may assume that $\eta = \infty$. In other words, we study recurrences of the type

$$z_{n+1} = \lambda_n z_n + b_n \qquad (b_n = \zeta_n(1 - \lambda_n);\, n \in \mathbb{N}) \tag{5.3}$$

(where ζ_n is the finite fixpoint of F_n) which is just a linear inhomogeneous recurrence of order one, and can be solved explicitly. In fact, as can easily be checked, its solutions are of the form

$$z_n = \lambda_{n-1} \cdot \cdots \cdot \lambda_0 \left(z_0 + \sum_{k=0}^{n-1} b_k (\lambda_k \cdot \cdots \cdot \lambda_0)^{-1} \right). \tag{5.4}$$

Using the identity $b_n = \zeta_n(1 - \lambda_n)$, we obtain an alternative expression for the solution

$$z_n = \zeta_n + \lambda_{n-1} \cdot \cdots \cdot \lambda_0 \left(z_0 - \zeta_0 + \sum_{k=0}^{n-1} (\lambda_k \cdot \cdots \cdot \lambda_0)^{-1} (\zeta_k - \zeta_{k+1}) \right). \tag{5.5}$$

In the remainder of this section, we assume that the fixpoints ζ_n of F_n are of bounded variation, i.e., that $\sum_{n=0}^{\infty} |\zeta_n - \zeta_{n+1}|$ converges. We shall see that the behaviour of the solutions of the recurrence $F_n(z_n) = z_{n+1}$ depends entirely on the products of the numbers $F_n(\zeta_n)$. We put together the different important cases in a theorem.

THEOREM 5.1. *Consider the recurrence*

$$F_n(z_n) = z_{n+1} \qquad (n \in \mathbb{N}), \tag{5.6}$$

where $F_n(z) = \lambda_n z + b_n$ *has fixpoints* ζ_n *and* ∞ $(n \in \mathbb{N})$. *If the sum* $\sum_{n=0}^{\infty} |\zeta_n - \zeta_{n+1}|$ *converges, then*

(1) *If* $\prod_{k=0}^{\infty} |F'_k(\zeta_k)| = 0$ *and* $\prod_{k=m}^{p} |F'_k(\zeta_k)|$ *is bounded from above for all* m, p *then all solutions* $\{z_n\} \neq \{\infty\}$ *of* (5.6) *converge to* $\zeta = \lim_{n \to \infty} \zeta_n$.

(2) *If* $\prod_{k=0}^{\infty} |F'_k(\zeta_k)| = \infty$ *and* $\prod_{k=m}^{p} |F'_k(\zeta_k)|$ *is bounded from below for all m, p, then all solutions of* (5.6) *converge to* ∞ *except for one solution that converges to* ζ.

(3) *If* $0 < m < \prod_{k=0}^{n} |F'_k(\zeta_k)| < M$ *for all n and real numbers m, M then* (5.6) *has one solution that converges to* ζ, *whereas all other "finite" solutions* $\{z_n\} \neq \{\infty\}$ *do not converge. Moreover, if* $\prod_{k=0}^{\infty} |F'_k(\zeta_k)|$ *converges, then all finite solutions* $\{z_n\}$ *converge to circles* $\{z \in \mathbb{C}: |z-\zeta| = r\}$ *for* $r \in \mathbb{R}$, *but they do not converge to single points, unless* $\prod_{k=0}^{\infty} F'_k(\zeta_k)$ *converges.*

Furthermore, in cases (2) *and* (3) *we have the equality*

$$z_n - \zeta_n = (\lambda_0 \cdot \cdots \cdot \lambda_{n-1}) \times \left(z_0 - \zeta_0 + C - \sum_{k=n}^{\infty} (\lambda_0 \cdot \cdots \cdot \lambda_k)^{-1} (\zeta_k - \zeta_{k+1}) \right) \tag{5.7}$$

for all $n \in \mathbb{N}$, *and for some complex number C.* (*The equality is also valid if the sequence* $\{\zeta_n\}$ *is not of bounded variation.*)

We shall use the following fact in order to prove the theorem:

LEMMA 5.2. *If* a_k *and* b_k *are sequences of complex numbers, such that* $\sum_{k=0}^{\infty} |b_k|$ *converges and* $|a_{n+k}/a_n| < M$ *for all n and* $k > 0$, *then* $\sum_{k=n}^{\infty} a_k b_k = o(a_n)$ *as* $n \to \infty$.

Proof. $|\sum_{k=n}^{\infty} a_k b_k| < M\, |a_n| \cdot \sum_{k=n}^{\infty} |b_k|$. Q.E.D

Proof of Theorem 5.1. (1) We use formula (5.5). Clearly, $\lambda_n = F'_n(\zeta_n)$. Let $A = \sup_{m,p} |\lambda_m \cdot \cdots \cdot \lambda_p|$. Take $\varepsilon > 0$ and N so large that $\sum_{k=N+1}^{\infty} |\zeta_k - \zeta_{k+1}| < \varepsilon/A$. Then, for $n \geqslant N$,

$$\left| (\lambda_0 \cdot \cdots \cdot \lambda_{n-1}) \sum_{k=0}^{n-1} (\lambda_0 \cdot \cdots \cdot \lambda_k)^{-1} (\zeta_k - \zeta_{k+1}) \right| \leqslant |\lambda_0 \cdot \cdots \cdot \lambda_{n-1}| \left(\sum_{k=0}^{N} |\lambda_0 \cdot \cdots \cdot \lambda_k|^{-1} \cdot |\zeta_k - \zeta_{k+1}| \right) + A \cdot \sum_{k=N+1}^{n-1} |\zeta_k - \zeta_{k+1}|.$$

The first term on the right side tends to zero as $n \to \infty$, and the second term is smaller than ε. From this and (5.5) it follows that $z_n - \zeta_n$ tends to zero as $n \to \infty$.

(2), (3) In these cases, the sum $\sum_{k=0}^{n-1} (\lambda_0 \cdot \cdots \cdot \lambda_k)^{-1} (\zeta_k - \zeta_{k+1})$ converges to some number $C \in \mathbb{C}$. Thus, formula (5.7) follows immediately from (5.5). If we choose $z_0 = \zeta_0 - C$, then

$$z_n-\zeta_n=-(\lambda_0\cdot\cdots\cdot\lambda_{n-1})\cdot\sum_{k=n}^{\infty}(\lambda_0\cdot\cdots\cdot\lambda_k)^{-1}(\zeta_k-\zeta_{k+1})$$
$$=\sum_{k=n}^{\infty}(\lambda_{k+1}\cdot\cdots\cdot\lambda_{n-1})(\zeta_k-\zeta_{k+1}).$$

In case (2), we use Lemma 5.2 for $a_k=(\lambda_0\cdot\cdots\cdot\lambda_k)^{-1}$ and $b_k=\zeta_k-\zeta_{k+1}$ which shows us that $z_n-\zeta_n$ tends to zero as $n\to\infty$. In case (3), we have $|\lambda_0\cdot\cdots\cdot\lambda_{n-1}|<M$, so that $z_n-\zeta_n$ tends to zero as $n\to\infty$. On the other hand, if we choose $z_0\neq\zeta_0-C$, we see that $z_n-\zeta_n=(\prod_{k=0}^{n-1}\lambda_k)(c+o(1))$ for $c\neq 0$. From this fact, it follows that $z_n-\zeta_n$ converges to ∞ in case (2), and does not converge in case (3), unless $\prod_{n=0}^{\infty}\lambda_n<\infty$. If $\prod_{n=0}^{\infty}|\lambda_n|<\infty$, then $z_n-\zeta_n$ converges to the circle $\{z\in\mathbb{C}\colon |z|=c\prod_{n=0}^{\infty}|\lambda_n|\}$. Q.E.D

The section of examples (Section 9) shows that things are not so easy if the fixpoints are not of bounded variation. In that case, it may happen that none of the solutions converge (except for the constant solution $\{\infty\}$), or that only some of them converge.

Comparing the results of Theorem 5.1 with the behaviour of the solutions of the "constant recurrence" $F(z_n)=z_{n+1}$, as discussed in the beginning of Section 3, we see that in cases (1) and (2) the solutions display "hyperbolic" behaviour, with ζ the attracting limit point in case (1) and the repelling limit point in case (2), whereas in case (3) the solutions display "elliptic" behaviour.

6. STABILITY

Let $\{F_n\}$ be a sequence of Möbius-transformations with converging fixpoints. This section is concerned with the case that at least one of the limits ζ of fixpoints is stable, i.e., that solutions $\{z_n\}$ of the recurrence $F_n(z_n)=z_{n+1}$ that are close to ζ for n large enough, remain so. For example, this phenomenon can be observed in cases (1) and (2) of Theorem 5.1 for one of the limits of fixpoints, and in case (3) for both. The aim of this section is to show that when this situation—that will be defined more precisely—occurs, then there is indeed a solution that converges to the limit point ζ. Finally, we shall see that although some solutions converge to ζ, this will in general not be the case for all solutions. To begin with, we define what is meant by stability.

Definition. Let $\{F_n\}$ be a sequence of Möbius-transformations, and $\zeta\in\mathbb{P}^1(\mathbb{C})$. A neighbourhood basis $\{U_\alpha\}_\alpha$ of ζ is called *stable* (*under* $\{F_n\}$) if for each U_α there is a number $n(\alpha)$ and a neighbourhood $V_\alpha\supset U_\alpha$ such that for any sequence $\{U_h\}_h$ converging to ζ, the corresponding sequence

$\{V_h\}_h$ tends to ζ as well and such that for any solution $\{z_n\}$ of the recurrence $F_n(z_n)=z_{n+1}$, if $z_m \in U_\alpha$ for some $m \geqslant n(\alpha)$, then $z_n \in V_\alpha$ for all $n \geqslant m$.

A point $\zeta \in \mathbb{P}^1(\mathbb{C})$ is called *stable* with respect to a recurrence $F_n(z_n)=z_{n+1}$ (or a sequence of Möbius-transformations $\{F_n\}$) if it has a stable neighbourhood basis under $\{F_n\}$.

Notice that there is no loss of generality if we take $U_\alpha=\{z \in \mathbb{P}^1(\mathbb{C}): d(z,\zeta)<\alpha\}$ for $\alpha>0$. We can now state the main result of this section.

THEOREM 6.1. *Suppose that $\zeta \in \mathbb{P}^1(\mathbb{C})$ is stable with respect to the sequence of Möbius-transformations $\{F_n\}$. Then the recurrence*

$$F_n(z_n)=z_{n+1} \qquad (n \in \mathbb{N}) \tag{6.1}$$

has a solution that converges to ζ, and for every neighbourhood U in the stable neighbourhood basis there is some solution $\{w_n\}$ such that $w_n \notin U$ for all n large enough.

Before we arrive at the proof of this result, we first prove another result, which is in itself not without interest. It says essentially that if the recurrence (6.1) has for every neighbourhood of some point $\zeta \in \mathbb{P}^1(\mathbb{C})$ a solution that remains in this neighbourhood from a certain index on then there is a solution that converges to ζ, provided that there is also a solution that comes not too close to ζ.

THEOREM 6.2. *Suppose that for some $\zeta \in \mathbb{P}^1(\mathbb{C})$ for each $\varepsilon>0$ the recurrence* (6.1) *has a solution $\{z_n\}=\{z_n(\varepsilon)\}$ such that $d(\zeta, z_n)<\varepsilon$ for all $n \geqslant N(\varepsilon)$, and furthermore that there exists some number $\varepsilon_0>0$ and a solution $\{w_n\}$ of* (6.1) *such that $d(\zeta, w_n)>\varepsilon_0$ for all $n \geqslant N$. Then* (6.1) *has a solution that converges to ζ. Moreover for each n, w_n is the limit point of a sequence $\{z_n(\varepsilon_i)\}_i$ such that $\varepsilon_i \to 0$, if and only if all solutions except for $\{w_n\}$ tend to ζ.*

Proof. Without loss of generality we may take $N=0$ and $\zeta=0$. In that case we can use $|z|$ instead of $d(z,\zeta)$ for $z \in \mathbb{C}$. Put $G_n(z)=z/(zw_n^{-1}+1)$ $(n \in \mathbb{N})$. From $|z|<\varepsilon<\varepsilon_0$ it follows that $|G_n(z)|<\varepsilon/(1-\varepsilon/\varepsilon_0)$ and if $|G_n(z)|<\varepsilon$, then $|z|=|G_n^{-1}(G_n(z))|<\varepsilon/(1-\varepsilon/\varepsilon_0)$. In particular, $z_n \to 0$ if and only if $G_n(z_n) \to 0$ $(n \to \infty)$. The recurrence $G_{n+1}^{-1}F_nG_n(z_n)=z_{n+1}$ has a constant solution $\{\infty\}$, so it is an inhomogeneous linear first-order recurrence. By (5.4), its other solutions have the form

$$z_n=(\lambda_0 \cdot \cdots \cdot \lambda_{n-1})(z_0+\Gamma_n) \qquad (n \in \mathbb{N}). \tag{6.2}$$

We now have sequences $\{\varepsilon_i\}$ and $\{c_i\}$ such that $\varepsilon_i \to 0$ as $i \to \infty$ and

$$|(\lambda_0 \cdot \cdots \cdot \lambda_{n-1})(c_i + \Gamma_n)| < \varepsilon_i \qquad \text{for} \quad n \geqslant N(\varepsilon_i). \tag{6.3}$$

Let c be a limit point of the sequence of the c_i's. There are two possibilities:

(1) c is finite. Then take $N'(\varepsilon_i)$ the minimum over the $N(\varepsilon_j)$ where $\varepsilon_j \leqslant \varepsilon_i$ and $|c - c_j| \leqslant 3/2\, |c_i - c_j|$. This implies that

$$|(\lambda_0 \cdot \cdots \cdot \lambda_{n-1})(c + \Gamma_n)| < \varepsilon_i + |(\lambda_0 \cdot \cdots \cdot \lambda_{n-1})(c - c_i)| < 4\varepsilon_i$$

for $n \geqslant N'(\varepsilon_i)$. Hence, the solution corresponding to c in the expression (6.2) converges to 0.

(2) $c = \infty$. The solution corresponding to the value $c = \infty$ is $\{w_n\}$ itself, which of course never can converge to 0. Inequality (6.3) implies that

$$|\lambda_0 \cdot \cdots \cdot \lambda_{n-1}| < \frac{\varepsilon_i + \varepsilon_j}{|c_i - c_j|} \qquad \text{for} \quad n \geqslant \max(N(\varepsilon_i), N(\varepsilon_j)),$$

and the term on the right-hand side tends to zero as $j \to \infty$, hence $\lambda_0 \cdot \cdots \cdot \lambda_{n-1}$ must tend to zero as well. But in that case,

$$|\Gamma_n(\lambda_0 \cdot \cdots \cdot \lambda_{n-1})| < \varepsilon_i + |c_i(\lambda_0 \cdot \cdots \cdot \lambda_{n-1})| < 2\varepsilon_i$$

for n large enough. Hence every solution distinct from $\{w_n\}$ must converge to zero. Q.E.D

As Example 9.6 in Section 9 will show, the existence of the solution $\{w_n\}$ that remains "far from" ζ is really necessary.

We use this result for the proof of Theorem 6.1:

Proof of Theorem 6.1. Since ζ has a stable neighbourhood basis with respect to the $\{F_n\}$, the recurrence (6.1) certainly has, for any neighbourhood of ζ, some solution that remains in that neighbourhood from a certain index on. In order to apply Theorem 6.2, we must show that there is some solution that does not approach ζ too closely.

Let $U = U_\alpha$ be one of the stable open sets belonging to this basis and $V \supset U$ the corresponding V_α. We may assume that the closure $\bar{V}$ of V is not the whole $\mathbb{P}^1(\mathbb{C})$. Put $N = n(\alpha)$. Let, for $n \geqslant N$, E_n be the set $U \cup F_N^{-1}(U) \cup \cdots \cup (F_{n-1} \cdots F_N)^{-1}(U)$. Clearly $E_N = U \subset V$, and $E_N \subset E_{N+1} \subset E_{N+2} \subset \cdots$ and $E_n \neq \mathbb{P}^1(\mathbb{C})$ for all n since otherwise for all solutions $\{z_n\}$ of (6.1), $z_n \in V$ for $n \geqslant n_0$ for some n_0, which is absurd. Further, all E_n are open sets. Hence, for the complements $E_n^c \supset E_{n+1}^c \supset \cdots$, which form a decreasing sequence of closed sets, the intersection $\bigcap_{n=N}^\infty E_n^c$ is not empty. Take some solution $\{w_n\}$ of (6.1) with $w_N \in \bigcap_{n=N}^\infty E_n^c$. Then $w_n \notin U$ for all $n \geqslant N$.

Q.E.D

Combining Theorem 6.1 and Corollary 4.2 we get the following result:

COROLLARY 6.3. *If* $\zeta \in \mathbb{P}^1(\mathbb{C})$ *is a stable attracting point of the recurrence* (6.1), *then all solutions except one of* (6.3) *converge to* ζ. *Moreover, if the stable neighbourhood basis extends to* $\mathbb{P}^1(\mathbb{C})\backslash\{\eta\}$, *then the remaining solution converges to* η.

As an example, we prove a simplified, non-quantitative, two-dimensional variant of Theorem 2.1 of [6]. (The result we prove here is largely a special case of the theorem mentioned but it is only for the sake of showing how the results of this section can be applied that the example below is given.)

EXAMPLE 6.4. Let a sequence of Möbius-transformations F_n be given by

$$F_n(z) = \frac{r_n z + p_n}{q_n z + s_n} \qquad (n \in \mathbb{N}), \tag{6.4}$$

where

$$|r_n/s_n| < 1, \qquad \sum_{n=0}^{\infty} (|s_n| - |r_n|) = \infty, \qquad \lim_{n \to \infty} \frac{|p_n| + |q_n|}{|s_n| - |r_n|} = 0.$$

Then all solutions $\{z_n\}$ of (6.1) except one converge to 0, whereas the remaining solution converges to ∞. If $|q_n|/(|s_n| - |r_n|)$ is only bounded, then all solutions except one converge to 0.

Proof. It suffices to show that $z = 0$ has a stable neighbourhood basis (whose sets U_α cover $\mathbb{C}$ in case $|q_n|/(|s_n| - |r_n|) \to 0$) and that every solution close to $z = 0$ converges to 0.

Choose $0 < \varepsilon < \infty$ arbitrarily. Then, if $|z_n| < \varepsilon$,

$$|z_{n+1}| \leqslant \frac{|r_n|\,\varepsilon + |p_n|}{|s_n| - |q_n|\,\varepsilon} < \varepsilon$$

provided that $|q_n|\,\varepsilon < |s_n|$ and $\zeta_n < \varepsilon < \eta_n$, with ζ_n, η_n the zeros of $|q_n|\,X^2 - (|s_n| - |r_n|)\,X + |p_n|$, where ζ_n converges to 0. In the case that $|q_n|/(|s_n| - |r_n|) \to 0$ the second zero η_n converges to ∞, in the more general case only $|\eta_n| > M > 0$ for n large enough. If $|z_n| \leqslant \zeta_n$, and $2\,|q_n|\,\varepsilon < |s_n|$, then

$$|z_{n+1}| \leqslant \frac{|r_n|\,|z_n| + |p_n|}{|s_n| - |q_n|\,|z_n|} < \frac{2(|r_n|\,|z_n| + |p_n|)}{|s_n|} < 2\zeta_n + 2\,|p_n|/|s_n|,$$

where the right-hand side tends to zero. Hence, we have a stable neighbourhood basis of $z=0$ which extends to $\mathbb{C}$ if $|q_n|/(|s_n|-|r_n|)\to 0$. (For $U_\alpha=\{z\in\mathbb{C}: |z|<\varepsilon\}$, we have

$$V_\alpha=\{z\in\mathbb{C}: |z|<\max_{n\geqslant N}(2\zeta_n+2\,|p_n|/|s_n|, \varepsilon)\}$$

for $N=n(\alpha)$ so large that all conditions $2|q_n|\,\varepsilon<|s_n|$, $\zeta_n<\varepsilon<\eta_n$ are satisfied for $n\geqslant N$.) Hence by Theorem 6.1 and the remark below Theorem 6.2, there is a solution that converges to 0. We show that all solutions close to $z=0$ converge to 0. Take $0<\varepsilon<1$, and $\varepsilon'=\sqrt{\varepsilon}<|z_n|<1$. Let N be so large that $|p_n|+|q_n|<\varepsilon(|s_n|-|r_n|)$ for $n\geqslant N$. Then

$$|z_{n+1}|\leqslant\frac{|r_n|\,|z_n|+\varepsilon(|s_n|-|r_n|)}{|s_n|-\varepsilon(|s_n|-|r_n|)\,|z_n|}$$

$$\leqslant\frac{|r_n|(1-\varepsilon')+\varepsilon'\,|s_n|}{\varepsilon'\,|r_n|+|s_n|(1-\varepsilon')}\cdot|z_n|$$

and since $\prod_{n=0}^{\infty}|r_n/s_n|=0$, also $\prod_{n=0}^{\infty}((|r_n|\,(1-\varepsilon')+\varepsilon'\,|s_n|)/(\varepsilon'\,|r_n|+|s_n|\,(1-\varepsilon')))=0$, so that in the end $|z_n|$ becomes arbitrarily small. We can now use Corollary 6.3 to show that in fact all solutions except one converge to $z=0$. Q.E.D

COROLLARY 6.5. *If the sequence of Möbius-transformations* $\{F_n\}$ *converges to some Möbius-transformation F which is either hyperbolic or loxodromic, then all solutions of the corresponding recurrence* (6.1) *converge to one of the fixpoints of F, one solution converging to the fixpoint* η *with* $|F'(\eta)|>1$, *the other solutions converging to the other fixpoint* ζ.

Proof. It suffices to take $\zeta=0$ and $\eta=\infty$. Then $F(z)=\lambda z$ with $|\lambda|<1$. Now the above example (or Theorem 2.1 of [6]) can be applied. Q.E.D

Mandell and Magnus [8] already showed that in this case all solutions except one converge to ζ. In fact, this result also follows from the matrix version of the Poincaré–Perron Theorem (as can be seen if we apply Theorem 1.1 for $k=2$).

7. SEQUENCES OF MÖBIUS-TRANSFORMATIONS WHOSE FIXPOINTS ARE OF BOUNDED VARIATION

The aim of this section is to show that if $\{F_n\}$ is a sequence of Möbius-transformations, and the fixpoints ζ_n and η_n of the F_n are of bounded variation (i.e., the sums $\sum_{n=0}^{\infty} d(\zeta_n, \zeta_{n+1})$ and $\sum_{n=0}^{\infty} d(\eta_n, \eta_{n+1})$ converge) and converge to distinct limits, then the behaviour of the solutions of the recurrence

$$F_n(z_n) = z_{n+1} \qquad (n \in \mathbb{N}) \tag{7.1}$$

behave as if the fixpoints ζ_n, η_n were constant for all n, i.e., the convergence behaviour depends entirely on $\prod_{n=0}^{\infty} |F'_n(\zeta_n)|$, at least if the product is bounded or ∞. In accordance with the convention mentioned in the Introduction, this means that all products $\prod_{n=m}^{p} |F'_n(\zeta_n)|$ are bounded either from above or from below (see Example 9.1).

THEOREM 7.1. *Let $\{F_n\}$ be a given sequence of Möbius-transformations whose fixpoints ζ_n and η_n are of bounded variation and converge to distinct points ζ and η in $\mathbb{P}^1(\mathbb{C})$. Then*

(1) *If $\prod_{n=0}^{\infty} |F'_n(\zeta_n)| = 0$, then all solutions but one converge to ζ, whereas the remaining solution converges to η.*

(2) *If $\prod_{n=0}^{\infty} |F'_n(\zeta_n)| = \infty$, then all solutions but one converge to η, whereas the remaining solution converges to ζ.*

(3) *If $0 < m < \prod_{n=m}^{p} |F'_n(\zeta_n)| < M$ for all m, p, there is exactly one solution that converges to ζ, and one solution that converges to η. If $\prod_{n=0}^{\infty} |F'(\zeta_n)|$ converges, then all the other solutions converge to circles $\{z \in \mathbb{P}^1(\mathbb{C}): |(z-\zeta)/(z-\eta)| = c\}$ for $c \in \mathbb{C}$, $c \neq 0$. If also $\prod_{n=0}^{\infty} F'_n(\zeta_n)$ converges, then all solutions converge to distinct points.*

Gill [4] proved the following result, which appears to be a special case of Theorem 7.1:

THEOREM. *If $\{F_n\}$ is a sequence of Möbius-transformations that converges to an elliptic map F, and if the fixpoints ζ_n, η_n of the F_n are of bounded variation and $|F'_n(\zeta_n)| < 1$ for all n, then all solutions except for at most one converge to $\zeta = \lim_{n \to \infty} \zeta_n$ if $\prod_{n=0}^{\infty} |F'_n(\zeta_n)| = 0$, whereas if $\prod_{n=0}^{\infty} |F'_n(\zeta_n)|$ converges, then there are two solutions that converge to the limits of fixpoints ζ and η, whereas the other solutions do not converge.*

We shall need Theorem 1.4. For simplicity, we restate Theorem 1.4 in the simplified version that we shall use here (for $k = 2$, and without the estimations).

LEMMA 7.2. *Let $\{\lambda_n\}$ be a sequence of complex numbers such that $\prod_{n=m}^{p} |\lambda_n|$ is bounded either from below or from above. Let*

$$M_n = \operatorname{diag}(\lambda_n, 1) + D_n \qquad (n \in \mathbb{N}),$$

where $\sum_{n=0}^{\infty} \|D_n\|/\max(1, |\lambda_n|) < \infty$. *Then there exists some sequence of matrices* $\{J_n\}$, *converging to identity, and some sequence of numbers* $\{\delta_n\}$, $\delta_n = O(\|D_n\|)$, *such that*

$$J_{n+1}^{-1} M_n J_n = \operatorname{diag}(\lambda_n + \delta_n, 1) \qquad (n \in \mathbb{N}).$$

If $\sum_{n=0}^{\infty} \|D_n\|/\min(1, |\lambda_n|)$ *converges, then we may take* $\delta_n = 0$.

Proof of Theorem 7.1. By (5.1), we have

$$(F_n(z) - \zeta_n)^{-1} = F'_n(\eta_n)(z - \zeta_n)^{-1} + (1 - F'_n(\eta_n))(\eta_n - \zeta_n)^{-1} \qquad (n \in \mathbb{N})$$

whence, by $F'_n(\zeta_n)\, F'_n(\eta_n) = 1$ (Lemma 3.1(5)),

$$F_n(z) - \zeta_n = \frac{F'_n(\zeta_n)(z - \zeta_n)}{1 + (F'_n(\zeta_n) - 1)(\eta_n - \zeta_n)^{-1}\,(z - \zeta_n)}$$

so that, for $\{z_n\}$ any solution of (7.1), $\{y_n\} = \{z_n - \zeta_n\}$ satisfies

$$\begin{aligned} y_{n+1} &= \frac{F'_n(\zeta_n)\, y_n}{1 + (F'_n(\zeta_n) - 1)(\eta_n - \zeta_n)^{-1}\, y_n} + \zeta_n - \zeta_{n+1} \\ &=: F_n^{(0)}(y_n) + \zeta_n - \zeta_{n+1}. \end{aligned} \tag{7.2}$$

For simplicity, we assume $\zeta = 0$, $\eta = \infty$. The recurrence $x_{n+1} = F_n^{(0)}(x_n)$ $(n \in \mathbb{N})$ has, by Theorem 5.1, solutions $\{0\}$ and $\{g_n\}$ with $\lim_{n \to \infty} g_n = \infty$ in either of the cases (1)–(3). In order to see this most easily, notice that

$$x_{n+1}^{-1} = F'_n(\eta_n)\, x_n^{-1} + (1 - F'_n(\eta_n))(\eta_n - \zeta_n)^{-1}.$$

Set $G_n(z) = z/(1 - z g_n^{-1})$ for $n \in \mathbb{N}$. Then $G_{n+1} F_n^{(0)} G_n^{-1}$ has fixpoints 0 and ∞, and $(G_{n+1} F_n^{(0)} G_n^{-1})'\,(0) = (F_n^{(0)})'\,(0) = F'_n(\zeta_n)$ hence $G_{n+1} F_n^{(0)} G_n^{-1}(z) = F'_n(\zeta_n) z$. In order to apply Lemma 7.2, we use the corresponding matrices: Set

$$M_n^{(0)} = \begin{pmatrix} F'_n(\zeta_n) & 0 \\ (F'_n(\zeta_n) - 1)(\eta_n - \zeta_n)^{-1} & 1 \end{pmatrix},$$

$$M_n = M_n^{(0)} + (\zeta_n - \zeta_{n+1}) \begin{pmatrix} (F'_n(\zeta_n) - 1)(\eta_n - \zeta_n)^{-1} & 1 \\ 0 & 0 \end{pmatrix},$$

and

$$N_n = \begin{pmatrix} 1 & 0 \\ -g_n^{-1} & 1 \end{pmatrix}$$

the matrices corresponding to $F_n^{(0)}$, $F_n(z)-\zeta_n$, and G_n, respectively. Then

$$N_{n+1}M_nN_n^{-1}(z)=\operatorname{diag}(F'_n(\zeta_n), 1)+D_n \qquad (n\in\mathbb{N})$$

with $\|D_n\|=O(|\zeta_n-\zeta_{n+1}|\max(1, |F'_n(\zeta_n|))$ since $N_n\to I$ as $n\to\infty$. Applying Lemma 7.2 and translating the result back into a result for Möbius-transformations we obtain that there exists a sequence of Möbius-transformations $\{H_n\}$, converging to the identity and a sequence of numbers $\{\delta_n\}$ with $\sum_{n=0}^{\infty}|\delta_n|/\max(1, |F'_n(\zeta_n)|)<\infty$ such that

$$H_{n+1}F_nH_n^{-1}(z)=(F'_n(\zeta_n)+\delta_n)(z) \qquad (n\in\mathbb{N})$$

with $\delta_n=O(|\zeta_n-\zeta_{n+1}|)\max(1, |F'_n(\zeta_n|)$. Because $\{H_n(z_n)\}$ converges if and only if $\{z_n\}$ does so, it is sufficient to show that the statements of the theorem are true for

$$z_{n+1}=(F'_n(\zeta_n)+\delta_n)\,z_n. \tag{7.3}$$

If $\prod_{n=0}^{\infty}|F'_n(\zeta_n)|=0$, we must show that also $\prod_{n=0}^{\infty}(|F'_n(\zeta_n)|+\delta_n)=0$. But this is true, because $\sum_{n=0}^{\infty}(1-|F'_n(\zeta_n)|-\delta_n)=\sum_{n=0}^{\infty}(1-|F'_n(\zeta_n)|)$ up to a finite number, since $\sum_{n=0}^{\infty}|\delta_n|<\infty$. If $\prod_{n=0}^{\infty}|F'_n(\zeta_n)|=\infty$ or if $0<m<\prod_{n=0}^{\infty}|F'_n(\zeta_n)|<M$, we can even take $\delta_n=0$. This proves the result, because the solutions of (7.2) are of the form $z_n=C\cdot\prod_{h=0}^{n-1}(F'_h(\zeta_h)+\delta_h)$ for $C\in\mathbb{C}$ or $C=\infty$. Q.E.D

By Corollary 6.5, if the sequence $\{F_n\}$ converges to a hyperbolic or loxodromic map, there is convergence of all solutions, even if the fixpoints of the F_n are not of bounded variation. However, this is not true in general if the maps F_n converge to an elliptic or a parabolic limit, as will be shown in Examples 9.3, 9.4, and 9.5. (Obviously, we do not expect all solutions to converge in the elliptic case, but at most two of them. Still, it may happen that none of the solutions converge if the fixpoints of the F_n are not of bounded variation.)

8. PARABOLIC BEHAVIOUR

In this section we consider sequences of Möbius-transformations with fixpoints that converge to the same limit (or Möbius-transformations that are themselves parabolic). In this case, the asymptotic behaviour of the solutions is much more sensitive to the coefficients: for instance, in this case it is not sufficient anymore that the fixpoints are of bounded variation in order to get a neat convergence result. One way to treat this "parabolic case" is to try to separate the fixpoints by a suitable transformation: we look for maps G_n (as neat as possible) such that $G_{n+1}^{-1} F_n G_n(z)$ are Möbius-transformations that have fixpoints converging to distinct limits, and to which results like those in Sections 4–7 can be applied. We shall not give an example of this in this section, but the reader can see that exactly this idea is applied twice in the proof of Theorem 10.1 (where we deal with matrix recurrences and not with Möbius-transformations, but the idea remains the same). Another way to treat this "parabolic case" that comes to mind is to use a similar perturbation method as in the preceding section. We shall give an example of how one can proceed in this case, but we shall not go into the matter very deeply. A similar method was used to prove Theorem 5.1 of [6]. Still another point of view is to study the behaviour of orbits under a sequence of Möbius-transformations in the complex plane, somewhat similar to what we did in Section 6. We intend to study the special case here that the sequence of Möbius-transformations converges to a parabolic Möbius-transformation and that there is a region in $\mathbb{P}^1(\mathbb{C})$ that is stable under all the Möbius-transformations involved. This point of view was inspired by the following theorem of O. Perron (in [14]):

THEOREM. *Consider the linear recurrence of order two given by*

$$u_{n+2} - (2 - \eta_1(n))\, u_{n+1} + (1 - \eta_0(n))\, u_n \qquad (n \in \mathbb{N})$$

with $\lim_{n\to\infty} \eta_0(n) = \lim_{n\to\infty} \eta_1(n) = 0$ *and* $\eta_1(n) \geqslant 0$ *and* $\eta_0(n) - \eta_1(n) \geqslant 0$ *for all n. Then* $\lim_{n\to\infty} u_{n+1}/u_n = 1$ *for all non-zero solutions* $\{u_n\}$.

If we put $z_n = u_{n+1}/u_n - 1$ for all n this amounts to saying that all solutions $\{z_n\}$ of the recurrence

$$z_{n+1} = \frac{(1 - \eta_1(n))\, z_n + \eta_0(n) - \eta_1(n)}{1 + z_n} \qquad (n \in \mathbb{N})$$

converge to zero. Later we shall see that this result is a special case of Theorem 8.3, and that the condition $\eta_1(n) \geqslant 0$ can be replaced by $\eta_1(n) \in \mathbb{R}$. But first we prove the following result:

THEOREM 8.1. *Let* λ_n, δ_n, c_n *be complex numbers* $(n \in \mathbb{N})$ *such that* $\prod_{n=0}^{\infty} \lambda_n$ *converges,* $\sum_{n=0}^{\infty} |\delta_n \sum_{h=0}^{n} c_h|$ *converges, and* $|\sum_{n=0}^{\infty} c_n| = \infty$, *whereas* $c_n / \sum_{h=0}^{n-1} c_h$ *tends to* 0 *as* $n \to \infty$. *Then all solutions of the recurrence*

$$F_n(z_n) = z_{n+1} \qquad (n \in \mathbb{N}) \tag{8.1}$$

with F_n *given by*

$$F_n(z) = \frac{\lambda_n z + \delta_n}{c_n z + 1} \qquad (n \in \mathbb{N})$$

converge to $\zeta = 0$. *Moreover, there is one "subdominant" solution* $\{w_n\}$ *in the sense that* $w_n \cdot \sum_{h=0}^{n-1} c_h \to 0$ *as* $n \to \infty$, *whereas for the other solutions* $\{z_n\}$, $z_n \cdot \sum_{h=0}^{n-1} c_h \to 1$ *as* $n \to \infty$, *so that in particular* $\lim_{n\to\infty}(w_n/z_n) = 0$.

If we take in Theorem 8, $\lambda_n = 1$, $\delta_n = 0$, and $c_n = c \neq 0$ for all n, then $F_n = F$ is just a parabolic map with fixpoint 0. In this case, the solutions of the recurrence (8.1) are $\{w_n\} = \{0\}$, the subdominant solution, and $\{z_n\} = \{1/(a+cn)\}$ for $a \in \mathbb{C}$. If we let $\sum_{n=0}^{\infty} |c_n|$ converge, then by Lemma 7.2 there exist Möbius-transformations G_n, converging to the identity map, such that $G_{n+1}^{-1} F_n G_n(z) = z$, so that all solutions of the recurrence converge to distinct limits, and if we let $\prod_n \lambda_n = 0$, and either $\sum_n |c_n|$ converges, or $|c_n|/(1-|\lambda_n|)$ is bounded, then by Example 6.4 or by Theorem 2.1 of [6] all solutions except one of the recurrence converge to $\zeta = 0$, whereas the one remaining solution does not converge to 0. This shows that the conditions on the coefficients of F_n cannot be weakened too much.

On the other hand, if we fix the sequences $\{\lambda_n\}$ and $\{c_n\}$, Theorem 8.1 essentially gives a condition on the sequence $\{\delta_n\}$ in order that the solutions of Theorem 8.1 are asymptotically equal to the solutions of the recurrence with the same λ_n, c_n but with $\delta_n = 0$ for all n. Here also, the condition on the δ_n is as sharp as possible (e.g., take $F_n(z) = (z + d/n(n+1))/(z+1)$ which has solutions of the form $\{a/n\}$ with a a zero of $X^2 - X - d$; see also Example 9.7).

Proof of Theorem 8.1. We use a perturbation method, as in Theorem 7.1. Let $F_n^{(0)}(z) = \lambda_n z/(1 + c_n z)$. The solutions of the recurrence

$$F_n^{(0)}(z_n) = z_{n+1} \qquad (n \in \mathbb{N})$$

are $\{w_n\} = \{0\}$ or $\{z_n\}$, where

$$z_n = \lambda_0 \cdot \cdots \cdot \lambda_{n-1} \left(z_0^{-1} + \sum_{h=0}^{n-1} c_h \lambda_0 \cdot \cdots \cdot \lambda_{h-1} \right)^{-1}.$$

Let $\{g_n\}$ be the solution with $g_0 = \infty$. Then

$$g_n^{-1} = \sum_{h=0}^{n-1} c_h (\lambda_h \cdot \cdots \cdot \lambda_{n-1})^{-1} \qquad (n \in \mathbb{N})$$

so that $g_n \cdot \sum_{h=0}^{n-1} c_h \to 1$ as $n \to \infty$.

Put $G_n(z) = g_n z/(1+z)$ for $n \geqslant N$ with N so large that $g_n \neq 0$ or ∞ for $n \geqslant N$. Then $G_n^{-1}(z) = z/(g_n - z)$ and $G_{n+1}^{-1} F_n^{(0)} G_n$ has fixpoints 0 and ∞, whereas $(G_{n+1}^{-1} F_n^{(0)} G_n)'\,(0) = (G_n'(0)/G_{n+1}'(0))\lambda_n$, hence

$$G_{n+1}^{-1} F_n^{(0)} G_n(z) = \lambda_n \frac{g_n}{g_{n+1}} z \qquad (n \geqslant N).$$

Furthermore, let $\tilde{F}_n$, $\tilde{G}_n$ be the matrices corresponding to F_n and G_n, respectively:

$$\tilde{F}_n = \begin{pmatrix} \lambda_n & \delta_n \\ c_n & 1 \end{pmatrix}, \qquad \tilde{G}_n = \begin{pmatrix} g_n & 0 \\ 1 & 1 \end{pmatrix}.$$

Then, for some $C > 0$,

$$\sum_{n=N}^{\infty} \|\tilde{G}_{n+1}^{-1} \tilde{F}_n \tilde{G}_n - \operatorname{diag}(\lambda_n g_n/g_{n+1}, 1)\| < C \cdot \sum_{n=N}^{\infty} |\delta_n|/|g_{n+1}| < \infty$$

by $g_n \sum_{h=0}^{n-1} c_h \to 1$. Thus, by Theorem 1.4 (or Lemma 7.2, which amounts to the same in this case), there exist Möbius-transformations H_n, converging to the identity map, such that

$$(G_{n+1} H_{n+1})^{-1} F_n G_n H_n(z) = \lambda_n \frac{g_n}{g_{n+1}} z.$$

Thus, (8.1) has solutions $\{z_n(c)\} = \{G_n H_n(c\lambda_0 \cdot \cdots \cdot \lambda_{n-1}/g_n)\}$ for $c \in \mathbb{C}$ and $\{G_n H_n(\infty)\}$ (which corresponds to $c = \infty$). For $c = 0$ we have $z_n(0) = g_n H_n(0)/(1 + H_n(0)) = o(g_n)$, and for $c \neq 0$ or $c = \infty$ we have $z_n(c)/g_n \to 1$ as $n \to \infty$. Q.E.D

Obviously, it is not essential that in Theorem 8.1 the fixpoints converge to 0. If the fixpoints converge to $\zeta \in \mathbb{P}^1(\mathbb{C})$, we can always apply a transformation and consider $G^{-1} F_n G$ instead of F_n such that the fixpoints of $G^{-1} F_n G$ converge to 0. In particular, if the F_n converge to some F in Theorem 8.1, F can be just any parabolic map. Notice that the solutions of the recurrence behave in a "hyperbolic" way. There is one subdominant solution, that corresponds to a solution converging to the repelling fixpoint; the other solutions, having all the same size, correspond to solutions converging to the attracting fixpoint. The limits of the two fixpoints coincide in this case. That this is not a mere metaphor is in fact shown by the proof, where we separate the fixpoints by some transformation, and where the resulting recurrence (defined by $G_{n+1}^{-1} F_n G_n$) is in fact of hyperbolic type. Similarly, there are recurrences of type (8.1) where the limit is parabolic and where the solutions behave "elliptically." An example is given in Example 9.7, where the set of initial values of the solutions that converge is the complement of a circle in $\mathbb{P}^1(C)$.

On the other hand, it may also happen that the maps F_n converge to some parabolic map F, and none of its solutions converge (see Example 9.5; compare also with the remark at the end of Section 7, and Example 9.3, where it is shown that a similar phenomenon may occur if F is elliptic). We conclude the first part of Section 8 with an application of Theorem 8.1. The following result was first proved by Gill [4].

EXAMPLE 8.2. Suppose that $\{F_n\}$ is a sequence of Möbius-transformations converging to some parabolic map F. Suppose that for the fixpoints ζ_n, η_n we have $\sum_{n=0}^{\infty} n\,|\zeta_n-\zeta_{n+1}|<\infty$ and $\sum_{n=0}^{\infty} |\eta_n-\zeta_n|<\infty$. Then all solutions of the recurrence $F_n(z_n)=z_{n+1}$ converge to the fixpoint ζ of F.

Proof. We use formula (7.2). Note that it is also valid if F_n is parabolic: in this case $F'_n(\zeta_n)=1$, so that η_n does not appear in the formula. Without loss of generality we take $\zeta=0$. In that case, $F(z)=z/(1+cz)$ for some $c\neq 0$. Formula (7.2) now shows that either F_n is parabolic or $(F'_n(\zeta_n)-1)(\eta_n-\zeta_n)^{-1}$ converges to c. In particular, $\sum_{n=0}^{\infty} |F'_n(\zeta_n)-1|$ converges, and Theorem 8.1 can be applied to formula (7.2) with

$$c_n=(-1+F'_n(\zeta_n))(\eta_n-\zeta_n)^{-1},$$

$$\lambda_n=F'_n(\zeta_n)+(\zeta_n-\zeta_{n+1})\,c_n,$$

$$\delta_n=\zeta_n-\zeta_{n+1}.$$

Note that $\prod_{n=0}^{\infty}\lambda_n$ converges since $\sum_{n=0}^{\infty}|\lambda_n-1|$ does. Q.E.D

Remark. In addition, Theorem 8.1 shows that for all solutions $\{z_n\}$ of the recurrence except one $\lim_{n\to\infty} nz_n=1/c$ whereas there is one solution $\{w_n\}$ with $nw_n=o(1)$.

We now proceed to the second result of this section.

DEFINITION. A region $U\subset\mathbb{P}^1(\mathbb{C})$ is called *stable under a sequence of Möbius-transformations* $\{F_n\}$ if $F_n(U)\subset U$ for all n.

DEFINITION A region $U\subset\mathbb{P}^1(\mathbb{C})$ is called *disk-like* if it is either the interior of a circle in $\mathbb{C}$, or union of the exterior of a circle and $\{\infty\}$ or a half-plane in $\mathbb{C}$.

The following result is easily seen to imply Perron's Theorem:

THEOREM 8.3. *Let H be a disk-like that is stable under a sequence of Möbius-transformations $\{F_n\}$ that converges to some parabolic map F, such that*

$F(H) \neq H$. Then all solutions of the recurrence (8.1) *converge to the fixpoint of F.*

Proof. We first show that ζ must lie on the boundary ∂H of H. By continuity, H must be stable under F (i.e., $F(H) \subset H$). Since F is parabolic, all solutions of the recurrence $F(z_n) = z_{n+1}$ converge to ζ, by Section 4. Hence, ζ must lie in the closure $\bar{H}$ of H, because solutions that enter H never leave it again. Since F^{-1} is also parabolic, and the complement $\bar{H}^c$ of $\bar{H}$ is stable under F^{-1}, ζ must lie in H^c as well. Hence, $\zeta \in \partial H$. Because of this, there is no loss of generality if we assume $\zeta = 0$ and $H = \{z \in \mathbb{C}\colon \Re z > 0\}$ (considering $G^{-1} F_n G$ instead of F_n for some suitable Möbius-transformation G). Then $F(z) = z/(1 + cz)$ and the stability of H implies that $\Re c > 0$ (by $\Re F(ai) > 0$ for some $a \in \mathbb{R}$). We show that: (1) Every solution that does not remain in an arbitrary neighbourhood U of ζ must eventually enter H. (2) Every solution that enters H must eventually enter some neighbourhood $V \subset U$ of ζ. (3) Every solution that enters $H \cap V$ will remain in $H \cap U$ forever. Notice that the first fact allows us to conclude that solutions that do not enter H also converge to ζ. Fix a small number $\varepsilon > 0$, and set $U = \{z \in \mathbb{C}\colon |z| \leqslant \varepsilon\}$ and $V = \{z \in \mathbb{C}\colon |z| \leqslant \varepsilon'\}$ for $\varepsilon' = \varepsilon (P/2)^Q$, for positive numbers $P \leqslant 1$ and Q whose values are determined below.

(1) Let $\{z_n\}$ be a solution of (8.1) and suppose that $|z_n| > \varepsilon$ for some large n, say $n \geqslant N$ (how large N must be will appear in the sequel). We show that $z_{n+m} \in H$ for m large enough. Firstly, setting $F_n(z) = (a_n z + b_n)/(c_n z + 1)$, we have

$$\Re \left(\frac{1}{c z_{n+1}} - \frac{1}{c z_n} \right) = \Re \frac{c_n z_n^2 + (1 - a_n) z_n - b_n}{c z_n (a_n z_n + b_n)}.$$

Since the expression on the right-hand side tends uniformly to 1 for $|z| > \varepsilon'$ as $n \to \infty$, we take N_0 so large that it is at least 1/2 for $|z_n| > \varepsilon'$ and $n \geqslant N_0$. Further, since $\Re c > 0$, there is some number M such that $|z| > \varepsilon'$ and $\Re(1/cz) > M$ implies $z \in H$. By $\Re(1/cz) > -1/\varepsilon\, |c|$ for $|z| > \varepsilon$ and $|z_n| > \varepsilon$ it suffices to show that $|z_{n+m}| > \varepsilon'$ as long as both $0 \leqslant m \leqslant Q = \lceil 2M + 2(\varepsilon\, |c|)^{-1} \rceil$ and $|z_{n+m}| \notin H$ in order to establish (1). Let $0 < P \leqslant 1$ be such that $|a_n - c_n z| > P$ for $n \geqslant N_0'$ and $|z| \notin H$. Further, let $N \geqslant \max(N_0, N_0')$ be so large that $|b_n| < \varepsilon'$ for $n \geqslant N$. Then, by $F_n^{-1}(z) = (z - b_n)/(a_n - c_n z)$, we have that, as long as $z_{n+1} \notin H$ and $|z_{n+1}| > \varepsilon'$,

$$|z_n / z_{n+1}| = |F_n^{-1}(z_{n+1}) / z_{n+1}| \leqslant (1 + |b_n| / |z_{n+1}|)/P < 2/P.$$

If, on the other hand, $z_{n+1} \notin H$ and $|z_{n+1}| < \varepsilon'$, then $|z| \leqslant 2\varepsilon'/P$. Thus, if $|z_n| > \varepsilon$, $z_n \notin H$ for $n \geqslant N$, then either $|z_{n+m}| > \varepsilon (P/2)^m \geqslant \varepsilon'$ or $z_{n+m} \in H$ for $0 \leqslant m \leqslant Q$.

(2) We show that solutions that enter H do indeed enter V. If $|z_n| \leqslant \varepsilon'$, then there is nothing to prove. If $|z_n| > \varepsilon'$ for $n \geqslant N$, then, as above, $\Re(1/cz_{n+1}) - \Re(1/cz_n) > 1/2$, so that for some $m \geqslant 0$ we have either $|z_{n+m}| \leqslant \varepsilon'$ or both $\Re(1/cz_{n+m}) \geqslant 0$ and $|z_{n+m}| > \varepsilon'$. In the former case, we are ready. In the latter case, we use that $|z| > \varepsilon'$, $\Re(1/cz) \geqslant 0$ implies $|1+cz| > 1+\varepsilon''$ for some $\varepsilon'' > 0$, hence $|1+c_n z| > 1+\varepsilon''/2$ for $n \geqslant N_1 \geqslant N$. This implies that for $\Re(1/cz) > 0$, $|z| > \varepsilon$ we have

$$|F_n(z)/z| \leqslant \left|\frac{a_n + b_n/z}{(c_n z + 1)}\right| < 1 - \varepsilon''/4$$

for $n \geqslant N_2$ large enough. From this inequality it follows that if $z_n \in H$, then indeed $z_{n+m} \in V$ for some $m \geqslant 0$.

(3) The only thing that can still go wrong at this stage is that z_n can leave V and become large before $\Re(1/cz_n) \geqslant 0$ again. But if $|z_n| > \varepsilon'$, then $\Re(1/cz_n) \geqslant -1/\varepsilon'|c|$, so that $\Re(1/cz_{n+m}) \geqslant 0$ for some $m \leqslant R = \ulcorner 2(\varepsilon' \, |c|)^{-1} \urcorner$. On the other hand, if $z \in H$, $|z| > \varepsilon'$, and n is large enough, then

$$|F_n(z)/z| \leqslant \left|\frac{a_n + b_n/z}{(c_n z + 1)}\right| < M'$$

for some M' because the denominator of the right-hand term is bounded from below for $z \in H$. Lastly, if $|z| \leqslant \varepsilon'$, and ε' is small enough, then $|F_n(z)| < 2\varepsilon'$, for n sufficiently large. Hence, if $|z_n| \leqslant \varepsilon'$ for some $n \geqslant N_3$, then $|z_{n+m}| < 2\varepsilon'(M')^R$ for all $m \geqslant 0$. Q.E.D

Let, for $n \in \mathbb{N}$, E_n be the set $\{z = z_0 \colon z_n \in H\}$ of initial values of solutions $\{z_n\}$ of (8.1) that lie in H from index n on. It is clear that $E_0 \supset E_1 \supset E_2 \ldots$ and all E_n are open sets in $\mathbb{P}^1(\mathbb{C})$, since $E_n = (F_{n-1} \cdots F_0)^{-1}(H)$ $(n \in \mathbb{N})$. The union of the sets E_n consists of the initial values of the solutions of (8.1) that enter H at some stage. The intersection $\bigcap_{n=0}^{\infty} E_n^c$ of the complements is closed and not empty, since E_n can never be the whole $\mathbb{P}^1(\mathbb{C})$ for any given n. This proves that there are solutions that never enter H—but we have seen that nevertheless these solutions converge to $\zeta = 0$—and the set $\mathscr{E}$ of their initial values is either a closed disk or a single point. In fact, both possibilities occur. For the case that $\mathscr{E}$ is a single point let, for all n, F_n be the constant parabolic map $F(z) = z/(1+cz)$ with $\Re c > 0$. It is clear that $H = \{z \in \mathbb{C} \colon \Re z > 0\}$ is stable under F. Moreover, all solutions are of the form $\{1/(a+cn)\}_n$ or $\{0\}$. It is obvious that all solutions enter H except for $\{0\}$, which remains on the boundary. See Example 9.8 for the case that $\mathscr{E}$ is a closed disk.

9. EXAMPLES

In this section we have collected a number of examples most of which are meant to show that the theorems we proved do not hold generally, but that the additional conditions we imposed are, in a sense, natural and cannot be broadened too much. For every example we shall refer to the section or the theorem to which they belong.

(1) The first example shows that our convention that $\prod_{n=0}^{\infty} \lambda_n = 0$ implies that $\prod_{n=m}^{p} \lambda_n$ is bounded from above for all m, p is a natural one. In fact, we have a recurrence that satisfies in all respects the conditions of Theorem 7.1(1), except that $\prod_{n=m}^{p} \lambda_n$ is not bounded for all m, p. We can see that Theorem 7.1, which says that all solutions must converge, is not valid. The example belongs to Theorem 7.1, but also to Theorem 1.4, because it shows that the condition on the boundedness of the numbers $\prod_{n=m}^{p} |a_i(n)/a_j(n)|$ is not a mere caprice of the proof. Consequently, it also belongs to Theorem 1.3.

EXAMPLE 9.1. Consider the recurrence

$$z_{n+1} = \lambda_n z_n + b_n \qquad (n \in \mathbb{N}), \tag{9.1}$$

where the numbers λ_n, b_n are defined as follows: Setting $m_i = e^{-1/i^2}$ and $v_i = i^3$ for i odd and $m_i = e^{1/i^2 - 2/i^3}$, $v_i = i^3$ for i even ($i \geqslant 1$) we let $\lambda_n = m_i$ for $n_{i-1} \leqslant n < n_i$ where $n_i = v_1 + \cdots + v_i$. Further, $b_n = 1/i^4$ for $n = n_i - 1$ (i odd) and $b_n = 0$ otherwise. Clearly, $\sum_{n=0}^{\infty} |b_n|$, $\sum_{n=0}^{\infty} |\lambda_n - \lambda_{n+1}|$ converge, and the fixpoints ∞ and $b_n/(1-\lambda_n)$ are of bounded variation. Then $m_i^{v_i} = e^{-i}$ for i odd and $m_i^{v_i} = e^{i-2}$ for i even, so that for $i \geqslant 0$

$$\lambda_{n_{2i+1}-1} \cdot \cdots \cdot \lambda_0 = m_1^{v_1} \cdot \cdots \cdot m_{2i+1}^{v_{2i+1}} = e^{-3i-1}$$

and for $i > 0$

$$\lambda_{n_{2i}-1} \cdot \cdots \cdot \lambda_0 = m_1^{v_1} \cdot \cdots \cdot m_{2i}^{v_{2i}} = e^{-i}.$$

This shows that $\prod_{n=0}^{\infty} \lambda_n = 0$, but that $\prod_{n=m}^{p} \lambda_n$ is not bounded. For $\{z_n\} \neq \{\infty\}$ a solution of (9.1) we have

$$z_n = \lambda_{n-1} \cdot \cdots \cdot \lambda_0 \left(z_0 + \sum_{h=0}^{n-1} b_h (\lambda_h \cdot \cdots \cdot \lambda_0)^{-1} \right).$$

Since the choice of z_0 obviously does not influence the convergence of the solution, we take $z_0 = 0$. Thus, for $j > 0$,

$$z_{2_{2j}} = e^{-j} \sum_{i=1}^{j} \frac{1}{(2i-1)^4} e^{3i-2} \geqslant \frac{1}{(2j-1)^4} e^{2j-2}$$

and the right-hand side clearly converges to ∞ as $j \to \infty$. On the other hand,

$$z_{n_{2j+1}} = e^{-3j-1} \sum_{i=1}^{j+1} \frac{1}{(2i-1)^4} e^{3i-2}$$

which converges to zero as $j \to \infty$. Namely, for $L < j$ fixed,

$$0 < z_{n_{2j+1}} < e^{-3j-1} \sum_{i=0}^{L-1} \frac{1}{(2i+1)^4} e^{3i+1} + \frac{1}{(2L+1)^4} \sum_{i=L}^{j} e^{3(i-j)},$$

where the first term on the right-hand side tends to zero as $j \to \infty$ and the second term is smaller than $2/(2L+1)^4$ for $j \geqslant L$. Thus we see that none of the solutions (except for $\{\infty\}$ which is fixed by construction) converges.

(2) The second example shows that it may happen that F_n converges and its fixpoints converge, but the solutions converge to points that are not equal to the limits of the fixpoints. By Lemma 4.1(4) this can only happen if the limit of the F_n is the identity map.

EXAMPLE 9.2. Let b_k, c_k be complex numbers such that $b_k \to 1$ and $c_k = o(b_k - 1)$ as $k \to \infty$. Define Möbius-transformations F_k, G_k by

$$F_k(z) = b_k z, \qquad G_k(z) = \frac{z + b_k c_k}{c_k z + b_k} \qquad (k \in \mathbb{N}).$$

Then F_k has fixpoints 0, ∞ and the fixpoints of G_k satisfy $c_k z^2 + (b_k - 1) z - b_k c_k = 0$, hence these converge to 0 and ∞ too. Further, $G_k F_k(z) = (z + c_k)/(c_k z + 1)$ has fixpoints 1 and -1 and $(G_k F_k)'(1) = (1 - c_k)/(1 + c_k)$. Since moreover $F_k \to \mathrm{id}$ we see that the recurrence $J_n(z_n) = z_{n+1}$ with $J_{2k} = F_k$, $J_{2k+1} = G_k$ $(k \geqslant 0)$ has two solutions that converge to 1 and -1. If we now take $c_k = 1/(k+1)$, then all solutions except one converge to 1, and if we choose $c_k = i/(k+1)$, then none of the solutions converge, except for the two that converge to 1 and -1. This follows from Theorem 7.1, but also from the following identity, which follows from Lemma 3.1(6):

$$\frac{z_{2n} - 1}{z_{2n} + 1} = \left(\prod_{h=0}^{n-1} (G_h F_h)'(1) \right) \frac{z_0 - 1}{z_0 + 1}.$$

(3) Next we give an example of a sequence of Möbius-transformations $\{F_n\}$ that converges to an elliptic map F, but where the corresponding recurrence $F_n(z_n) = z_{n+1}$ has no converging solutions (see Section 7).

EXAMPLE 9.3. Let a_k $(k \geqslant 0)$ be imaginary numbers, converging to 0, and such that $\sum_{n=0}^{\infty} ia_n = \infty$. Set $c_k = 2a_k/(a_k^2+1)$. Then $c_k \to 0$ as $k \to \infty$ and c_k is imaginary. Let

$$F_{2k}(z) = \frac{z - c_k}{c_k z - 1}, \qquad F_{2k+1}(z) = \frac{z + c_k}{-c_k z - 1} \qquad (k \geqslant 0).$$

Then F_{2k} has fixpoints a_k, a_k^{-1}, and $F'_{2k}(a_k) = -1$. Similarly, F_{2k+1} has fixpoints $-a_k$, $-a_k^{-1}$, and $F'_{2k+1}(-a_k) = -1$. Further,

$$H_k(z) = F_{2k+1} F_{2k}(z) = \frac{(1+c_k^2)\, z - 2c_k}{-2c_k z + (1+c_k^2)} \qquad (k \geqslant 0)$$

has fixpoints 1 and -1 and $H'_k(1) = ((1+c_k)/(1-c_k))^2 = ((1+a_k)/(1-a_k))^4$ so that $|H'_k(1)| = 1$ and $\prod_{k=0}^{\infty} H'_k(1)$ does not converge (here we use that $\sum_{n=0}^{\infty} ia_n = \infty$). Hence the recurrence $H_k(z_k) = z_{k+1}$ has only two converging solutions, $\{1\}$ and $\{-1\}$ (by Theorem 7.1 or by an argument similar to that used in Example 9.2).

Since the F_n converge to $F(z) = -z$ we see immediately that the recurrence $F_n(z_n) = z_{n+1}$ has no converging solutions at all. For the case that the Möbius-transformations converge to an arbitrary elliptic map $F(z) = \theta z$ with $|\theta| = 1$, consider the recurrence $J_n(z_n) = z_{n+1}$ $(n \in \mathbb{N})$ with J_n defined as follows.

Let $N_1, N_2, \ldots$ be natural numbers, and θ_k roots of unity such that $\theta_k^{N_k} = -1$ for all k and $\theta_k \to \theta$ as k increases. Let G_n be Möbius-transformations defined by

$$\frac{G_{2k}(z) - a_k}{G_{2k}(z) - a_k^{-1}} = \theta_k \frac{z - a_k}{z - a_k^{-1}}, \qquad \frac{G_{2k+1}(z) + a_k}{G_{2k+1}(z) + a_k^{-1}} = \theta_k \frac{z + a_k}{z + a_k^{-1}}.$$

Then $G_{2k}^{N_k} = F_{2k}$, $G_{2k+1}^{N_k} = F_{2k+1}$, and $G_n(z) \to \theta z$. Define $J_n = G_{2k}$ for $n = 2N_1 + \cdots + 2N_{k-1} + i$, $J_n = G_{2k+1}$ for $n = 2N_1 + \cdots + 2N_{k-1} + N_k + i$ $(0 < i < N_k)$. Then $J_n(z)$ converges to θz and $J_n(z_n) = z_{n+1}$ has no converging solutions.

(4) Here is another instance of a sequence of Möbius-transformations converging to some elliptic map. We present two variants: in one case all solutions except one converge, and the remaining solution converges to a circle. In the second case, all solutions except one do not converge to a point, but to a circle.

EXAMPLE 9.4 (A). Let $0<\phi<2\pi$ be some positive real number and $\xi\in\mathbb{C}$. Set $\lambda_n=(1+1/n)\,e^{i\phi}$, and $\xi_n=\xi+\sum_{h=n}^{\infty}(1/h)\,e^{i(h+1)\phi}$. Now consider the recurrence

$$z_{n+1}=F_n(z_n)=\lambda_n z_n+(1-\lambda_n)\,\xi_n \qquad (n>0).$$

F_n converges to $F(z)=e^{i\phi}z+(1-e^{i\phi})\,\xi$, which is an elliptic map. The general form of the solutions is

$$z_n=\xi_n+ne^{i\phi n}(c-1/n) \qquad (n>0),$$

where $c\in\mathbb{P}^1(\mathbb{C})$, as can easily be ascertained. For $c\neq 0$, z_n converges to ∞, but for $c=0$, $z_n=\xi_n-e^{i\phi n}$, which does not converge to a single limit, but converges to the circle $\{z\in\mathbb{C}\colon |z-\xi|=1\}$.

EXAMPLE 9.4 (B). Let $0<\phi<2\pi$ be some positive real number and $\xi\in\mathbb{C}$. Set $\lambda_n=ne^{i\phi}/(n+1)$, and $\xi_n=\xi+\sum_{h=n+1}^{\infty}(1/h)\,e^{ih\phi}$. Consider the recurrence

$$z_{n+1}=F_n(z_n)=\lambda_n z_n+(1-\lambda_n)\,\xi_n \qquad (n>0).$$

As in (A), F_n converges to $F(z)=e^{i\phi}z+(1-e^{i\phi})\,\xi$, which is an elliptic map. The general form of the solutions is

$$z_n=\xi_n+\frac{1}{n}\,e^{i\phi n}(c+n) \qquad (n>0),$$

where $c\in\mathbb{P}^1(\mathbb{C})$. For all $c\in\mathbb{C}$, $(z_n-\xi_n)\,e^{in\phi}$ converges to 1. In particular, all finite solutions converge to the circle $\{z\in\mathbb{C}\colon |z-\xi|=1\}$.

(5) An example similar to Example 9.3 can be given for a sequence of Möbius-transformations F_n converging to a parabolic map F, where none of the solutions of the corresponding recurrence converge (compare with Section 8).

EXAMPLE 9.5. Let $0<N_1<N_2\ldots$ be an increasing sequence of natural numbers. Let $a_k=\tan(\pi/2N_k)$ and $b_k=\tan(\pi/4N_k)$ $(k\geqslant 0)$. Then $(1-ia_k)/(1+ia_k)=e^{-\pi i/N_k}$, $(1-ib_k)/(1+ib_k)=e^{-\pi i/2N_k}$, and $(a_k/b_k)\to 2$ as $k\to\infty$. Define Möbius-transformations F_k, G_k by

$$F_k(z)=\frac{z+ia_k^2}{iz+1}, \qquad G_k(z)=\frac{z+ib_k^2}{iz+1} \qquad (k\in\mathbb{N}).$$

F_k has fixpoints a_k, $-a_k$ and $F'_k(a_k)=(1-ia_k)/(1+ia_k)$ so that $F_k^{N_k}(z)=a_k^2/z$. Similarly, G_k has fixpoints $\pm b_k$ and $G_k^{2N_k}(z)=b_k^2/z$. Hence, $F_k^{N_k}G_k^{2N_k}(z)=(a_k/b_k)^2\,z$ which converges to $4z$. Define the recurrence $J_n(z_n)=z_{n+1}$ by

$$J_n = G_k \qquad \text{for} \quad n = 3N_1 + \cdots + 3N_{k-1} + i,$$
$$J_n = F_k \qquad \text{for} \quad n = 3N_1 + \cdots + 3N_{k-1} + 2N_k + j$$

for $0 < i \leqslant 2N_k$ and $0 < j \leqslant N_k$. Then $J_n(z) \to z/(iz+1)$ where the limit is a parabolic map with fixpoint 0. Since $F_k^{N_k} G_k^{2N_k}(z) \to 4z$ $(k \to \infty)$ the solutions of the recurrence defined by the J_n can only converge to 0 and ∞, but $F_k^{N_k}$ interchanges 0 and ∞. Hence there can be no converging solutions.

(6) The next example is inspired by Theorem 6.2, which says that if for any neighbourhood of a point ζ the recurrence $F_n(z_n) = z_{n+1}$ has some solution that lies in that neighbourhood almost always (i.e., for all but a finite number of indices), then there is a solution of the recurrence that really converges to ζ provided that there is also some solution that lies almost always outside some neighbourhood of ζ. We show here that the latter condition cannot be dispensed with.

Example 9.6. Let, for $j \geqslant 1$, Möbius-transformations G_j and H_j be given by

$$G_j(z) = \frac{z-1}{1+jz}, \qquad H_j(z) = \frac{(j/(j+1))(z+1)}{1-jz}.$$

Then $H_j G_j(z) = (j/(j+1))\,z$ and $F_j(z) = G_j H_{j-1} G_{j-1} \cdots H_1 G_1(z) = (z/(j-1))/(1+z)$. $F_j(z)$ converges to $-1/(1+z)$. Let $\{\varepsilon_k\}$ be a decreasing sequence of numbers $0 < \varepsilon_k < 1$ that tends to 0 as $k \to \infty$, and define numbers $z_0^{(k)} = 1/\varepsilon_k - 1$ that are to be the initial values of solutions $\{z_n^{(k)}\}$ of the recurrence $J_n(z_n) = z_{n+1}$ with $J_{2h-1} = H_h$, $J_{2h-2} = G_h$ $(h > 0)$. Obviously, $z_{2n-1}^{(k)} = F_n(z_0^{(k)}) = (1-\varepsilon_k)/n - \varepsilon_k$ and $z_{2n}^{(k)} = z_0^{(k)}/(n+1)$ so that $|z_m^{(k)}| < \varepsilon_k$ for m large enough. But there is no solution that converges to 0: For suppose there is one, say $\{w_n\}$; then $w_{2n-1} = F_n(w_0) = (w_0/n - 1)/(1+w_0)$. This can only converge to 0 if $w_0 = \infty$. But in that case, $w_{2n} = w_0/(n+1) = \infty$, which contradicts the assumption that w_n converges to zero.

(7) Here we give an example of a sequence $\{F_n\}$ of Möbius-transformations with a stable region H and converging to a parabolic transformation F. The recurrence $F_n(z_n) = z_{n+1}$ has solutions that converge if and only if the initial values do not lie on a circle in $\mathbb{P}^1(\mathbb{C})$. Theorem 8.3 does not apply here because $F(H) = H$ here (and not a genuine subset of H). This shows that the condition that F may not map H onto itself cannot be dispensed with.

Example 9.7. Let $F_n(z) = (z + c/(n^2+n))/(z+1)$ $(n > 0)$. The recurrence

$$F_n(z_n) = z_{n+1} \qquad (n \geqslant 1)$$

has two solutions $\{a/n\}$ and $\{b/n\}$ with a, b the zeros of the polynomial $X^2 - X - c$, as can easily be ascertained. If c is real and $c < -1/4$, then a, b are not real and $b = \bar{a}$. It thus follows that, for $\{z_n\}$ a solution of the recurrence,

$$\frac{z_{n+1} - a/(n+1)}{z_{n+1} - b/(n+1)} = \alpha_n \frac{z_n - a/n}{z_n - b/n} \qquad (n > 0),$$

where $\alpha_n = (c/a - n)/(c/b - n)$ (this can be seen by calculating the image of, say, $z = 0$ under F_n). Hence,

$$\frac{z_n - a/n}{z_n - b/n} = C\alpha_{n-1} \cdot \cdots \cdot \alpha_0 = C\Theta_n \qquad (n > 0)$$

for some constant C, so that we find for the general solution

$$z_n = \frac{a - bC\Theta_n}{1 - C\Theta_n} \cdot \frac{1}{n} \qquad (n > 0). \tag{9.2}$$

Note that $|\Theta_n| = 1$ and that $\alpha_n - 1 = d/n(1 + o(1))$ for some imaginary number $d \neq 0$. Hence, Θ_n does not converge. Now consider the solutions themselves. For $|C| \neq 1$ the denominator in (9.2) is bounded from below, whence $z_n = O(1/n)$. On the other hand, if $|C| = 1$, then for infinitely many values of n, $|C^{-1} - \Theta_n| < C_1/n$ for some $C_1 > 0$ (this follows from the fact that $\alpha_n - 1 = d/n(1 + o(1))$). Hence z_n cannot converge in this case (but nevertheless it is "most of the time" very close to 0, as follows also from Lemma 4.1(1)). Notice that there are two stable regions: the upper and lower half-plane. The solutions that do not converge lie on the real axis.

(8) In Section 8, Theorem 8.3, we studied the case where the maps F_n converge to a parabolic map F and that there is a stable region H such that $F_n(H) \subset H$ for all n and $F(H) \neq H$. We saw that in this case all solutions of the recurrence defined by the maps F_n converge to the fixpoint of F, but that either there is exactly one solution that never enters H or the solutions that never enter H have their initial values in some closed disk. An example of the first case has already been given at the end of Section 8: take all F_n equal to F. We now give an example of the second case.

EXAMPLE 9.8. Let $c > 0$ be a given real number and set $\lambda_n = (n-1)/(n+1)$ and $F_n(z) = (\lambda_n z)/(1 + cz)$ $(n > 1)$. It is clear that F_n converges to a parabolic map and that the half-plane $H = \{z \in \mathbb{C} : \Re z > 0\}$ is stable under all F_n. Also, $F(H) \neq H$. Hence we are in the situation studied in the second part of

Section 8. It can easily be checked that the recurrence $F_n(z_n) = z_{n+1}$ $(n > 1)$ has solutions

$$z_n = \frac{1}{n(n-1)} \left(A - \frac{c}{n-1} \right)^{-1} \qquad (n > 1)$$

for $A \in \mathbb{P}^1(\mathbb{C})$ ($A = \infty$ corresponds to the fixed solution $\{z_n\} = \{0\}$). Hence $\Re z_n^{-1} = (\Re A) \cdot n(n-1) - cn$ and $\{z_n\}$ enters H precisely if $\Re A > 0$. The initial values $z_2 = \frac{1}{2}(A - c)^{-1}$ of the solutions that do not enter H lie in the complement of the region $\Re((1/2z) + c) > 0$ which is precisely the closed disk $\{z \in \mathbb{C}: |z + (1/4c)| \leqslant 1/4c\}$.

10. LINEAR SECOND-ORDER RECURRENCES

In this last section we apply some of the results obtained above to study the asymptotic behaviour of the solutions of linear second-order recurrences

$$u_{n+2} + p(n)\, u_{n+1} + q(n)\, u_n = 0 \qquad (n \in \mathbb{N}), \tag{10.1}$$

in particular those that have coefficients which are sums of fractional powers n^{-a_i} $(a_i \geqslant 0)$ of n plus a small perturbation term of order $O(n^{-2-\varepsilon})$. In particular, the results obtained will hold if the coefficients have asymptotic expressions which are power series in fractional powers of $1/n$.

We shall see that in this case, if $\lim_{n \to \infty} p(n) = p$ and $\lim_{n \to \infty} q(n) = q$ exist and p, q are not both zero, there are always solutions $\{u_n\}$ such that u_{n+1}/u_n converge to the zeros α_1, α_2 of the polynomial $X^2 + pX + q$. In the case that the zeros have distinct moduli, this is exactly the Poincaré–Perron Theorem (see the remark below Theorem 1.2).

The case that the eigenvalues are distinct (but possibly having equal moduli) is covered by Corollary 1.6, because in this case the zeros $\alpha_1(n)$, $\alpha_2(n)$ of the polynomials $X^2 + p(n)\, X + q(n)$ are also sums of fractional powers of $1/n$ plus some term of order $n^{-2-\varepsilon}$. Hence the conditions of Corollary 1.6 are satisfied. Because of this, we can limit ourselves to the case that the zeros of $\chi(X) = X^2 + pX + q$ are equal and non-zero. Before we proceed, we recall two important notions:

Definition. A non-zero solution $\{u_n\}$ of a linear recurrence (10.1) is *subdominant* if $\lim_{n \to \infty} u_n/v_n = 0$ for all solutions $\{v_n\}$ of (10.1) that are linearly independent with $\{u_n\}$.

Definition. A real, non-zero solution $\{u_n\}$ of a linear recurrence (10.1) *oscillates* if $u_n u_{n+1} \leqq 0$ for infinitely many n.

We shall see that in the case that the zeros of the characteristic polynomial are equal and non-zero, two different situations can occur:

(1) There are exactly two solutions $\{u_n^{(1)}\}$, $\{u_n^{(2)}\}$ such that the quotients $u_{n+1}^{(i)}/u_n^{(i)}$ converge to the zero α of χ, and $\lim_{n\to\infty} |u_n^{(1)}/u_n^{(2)}|=1$, but $\lim_{n\to\infty} u_n^{(1)}/u_n^{(2)}$ does not converge. Further, if $p(n), q(n)\in\mathbb{R}$, then all real solutions $\{u_n\}$ oscillate. This we shall call the elliptic case.

(2) For all non-zero solutions $\{u_n\}$ the quotients u_{n+1}/u_n converge to α and there is a subdominant solution $\{u_n^{(1)}\}$. This will be referred to as the hyperbolic case.

In order to make things more simple we shall apply a transformation to (10.1) so that the recurrence depends only on one variable sequence, but without losing information about the solutions of the original recurrence. This can be done in the following manner: Setting $v_n=u_n\prod_{h=N}^{n-2}(-2/p(h))$ and $C(n)=1-4q(n)/(p(n)\,p(n-1))$ for $n\geqslant N$, where N is so large that $p(n)\neq 0$ for $n\geqslant N-1$ we have that $\{u_n\}$ is a solution of (10.1) if and only if $\{v_n\}$ is a solution of

$$v_{n+2}-2v_{n+1}+(1-C(n))\,v_n \qquad (n\geqslant N). \tag{10.2}$$

It is clear that the type of the recurrence (elliptic or hyperbolic, as defined above) does not change if we replace (10.1) by (10.2). Moreover, if $p(n)$, $q(n)$ are converging power series in n^{-a}, the same holds for $C(n)$. Note that, in the case that the zeros of the characteristic polynomial of (10.1) are equal and non-zero, $\lim_{n\to\infty} C(n)=0$ and the zero of the characteristic polynomial X^2-2X+1 is 1. We have the following result:

THEOREM 10.1. *Let $C(n)=a_1 n^{j_1}+\cdots+a_k n^{j_k}+O(n^{-2-\varepsilon})$ for $a_1, ..., a_k\in\mathbb{C}$, $-2\leqslant j_k<\cdots<j_1<0$, and $\varepsilon>0$. Then the recurrence (10.2) is of hyperbolic type if*

(1) *$\lim_{n\to\infty} n^2C(n)=d\in\mathbb{C}$ and d is not a negative real number $<-1/4$. In this case, if $d\neq -1/4$, there are two solutions $\{v_n^{(1)}\}$, $\{v_n^{(2)}\}$ with*

$$\lim_{n\to\infty} n\left(\frac{v_{n+1}^{(i)}}{v_n^{(i)}}-1\right)=r_i \qquad (i=1,2) \tag{10.3a}$$

with r_1, r_2 the zeros of X^2-X-d.

Further, if $n^2C(n)\to d\neq -1/4$ and $\sum_{n=0}^{\infty} |nC(n)-d/n|$ converges, then there are solutions $\{v_n^{(1)}\}$, $\{v_n^{(2)}\}$ with $\lim_{n\to\infty} v_n^{(i)}/n^{r_i}=1$ $(i=1,2)$ with r_1, r_2 the roots of X^2-X-d. If $d=-1/4$, and $\sum_{n=0}^{\infty}\log n\,|nC(n)-d/n|$ converges, then there are solutions $\{v_n^{(1)}\}$, $\{v_n^{(2)}\}$ with $\lim_{n\to\infty} v_n^{(1)}/n^{1/2}=1$ and $\lim_{n\to\infty} v_n^{(2)}/n^{1/2}\log n=1$,

(2) $\sum_{n=0}^{\infty} \Re \sqrt{C(n)} = \infty$ *where the branch of the square root is chosen such that* $\Re \sqrt{C(n)} \geqslant 0$. *This is the case when either* a_1 *is not a negative real number or when* $a_1 < 0$ *and* $a_l \notin \mathbb{R}$ *for some* $1 < l \leqslant k$ *with* $j_l - j_1/2 \geqslant -1$. *In this case, we have solutions* $\{v_n^{(1)}\}$, $\{v_n^{(2)}\}$ *with*

$$\lim_{n\to\infty} \left(\frac{v_{n+1}^{(i)}}{v_n^{(i)}} - 1 \right) \cdot \frac{1}{\sqrt{C(n)}} = (-1)^i \qquad (i = 1, 2). \tag{10.3b}$$

In particular, $\lim_{n\to\infty} v_n^{(1)}/v_n^{(2)} = 0$.

In the remaining cases, (10.2) *is of elliptic type. In the case that* $n^2 C(n) \to \infty$, *there are two solutions* $\{v_n^{(1)}\}$, $\{v_n^{(2)}\}$ *such that* (10.3b) *holds. If* $n^2 C(n) \to d < -1/4$, *then there are two solutions* $\{v_n^{(1)}\}$, $\{v_n^{(2)}\}$ *such that* (10.3a) *is valid* (*with* r_1, r_2 *the zeros of* $X^2 - X - d$). *In both cases,* $\lim_{n\to\infty} v_n^{(1)}/v_n^{(2)}$ *does not converge, but* $\lim_{n\to\infty} |v_n^{(1)}/v_n^{(2)}|$ *converges*

If moreover $\sum_{n=0}^{\infty} |nC(n) - d/n|$ *converges, then there are solutions* $\{v_n^{(1)}\}$, $\{v_n^{(2)}\}$ *with* $\lim_{n\to\infty} v_n^{(i)}/n^{r_i} = 1$ $(i = 1, 2)$ *with* r_1, r_2 *the roots of* $X^2 - X - d$.

Lastly, if $C(n) \in \mathbb{R}$ *for all* n, *then the real solutions* $\{v_n\}$ *of* (10.2) *oscillate in the elliptic case, but not in the hyperbolic case.*

We need a simple, but useful lemma that connects the behaviour of u_{n+1}/u_n to the behaviour of u_n/v_n for $\{u_n\}$, $\{v_n\}$ solutions of a linear recurrence.

LEMMA 10.2. *Let* $\{u_n\}$, $\{v_n\}$ *be non-zero solutions of a linear second-order recurrence* (10.1). *For every non-zero solution* $\{w_n\}$ *there exists a constant* C *such that*

$$\frac{w_{n+1}/w_n - u_{n+1}/u_n}{w_{n+1}/w_n - v_{n+1}/v_n} = C \cdot \frac{v_n}{u_n}.$$

Proof. Subtracting $(u_{n+2} + p(n)\, u_{n+1} + q(n)\, u_n)\, w_{n+1} = 0$ from $(w_{n+2} + p(n)\, w_{n+1} + q(n)\, w_n)\, u_{n+1} = 0$, we obtain

$$u_{n+2} w_{n+1} - u_{n+1} w_{n+2} = q(n)(u_{n+1} w_n - u_n w_{n+1}) \qquad (n \in \mathbb{N})$$

so that

$$u_{n+1} w_n - w_{n+1} u_n = c \cdot \prod_{h=0}^{n-1} q(h) \qquad (n \in \mathbb{N}),$$

where c depends only on $\{w_n\}$, $\{u_n\}$. Hence,

$$\frac{w_{n+1}/w_n - u_{n+1}/u_n}{w_{n+1}/w_n - v_{n+1}/v_n} = \frac{w_{n+1} u_n - w_n u_{n+1}}{w_{n+1} v_n - w_n v_{n+1}} \cdot \frac{v_n}{u_n} = C \cdot \frac{v_n}{u_n}. \qquad \text{Q.E.D.}$$

Another result that we shall use in order to decide when real solutions of (10.2) oscillate is the following lemma.

LEMMA 10.3. *Consider the recurrence*

$$z_{n+1} = \frac{z_n + C(n)}{z_n + 1} \qquad (n \in \mathbb{N}) \tag{10.4}$$

with $C(n) \in \mathbb{R}$ *for all* n *and* $\lim_{n\to\infty} C(n) = 0$. *The solution* $\{z_n\}$ *are of the form* $z_n = v_{n+1}/v_n - 1$ *for* $\{v_n\} \neq \{0\}$ *solutions of* (10.2). *If* (10.4) *has a real solution that converges, then all solutions converge to* 0 *and* (10.2) *has a subdominant solution, but no oscillating solutions. If* (10.4) *has no converging solutions, then all real solutions of* (10.2) *oscillate.*

Proof. Equation (10.4) is equivalent to

$$z_n z_{n+1} + z_{n+1} - z_n - C(n) = 0$$

and if we set $z_n = v_{n+1}/v_n - 1$, then it follows immediately that $\{v_n\}$ is a solution of (10.2) and conversely. If $\{z_n\}$ is real and converges, it can only converge to 0. The fact that all solutions converge in that case, and that (10.2) has a subdominant solution, is Proposition 2.2 and Corollary 7.2 of [7]. Suppose that the real solutions of (10.4) do not converge. Since $z_{n+1} \leqslant z_n$ unless either $z_n^2 < C(n)$ or $z_n \leqslant -1$ (or $z_n = \infty$, which implies $z_{n-1} = -1$), it follows that we must have $z_n \leqslant -1$ for infinitely many n. If $z_n < -1$, then $v_{n+1}/v_n < 0$ for the corresponding solution $\{v_n\}$ of (10.2), hence v_n and v_{n+1} have unequal signs. If $z_n = -1$, then $v_{n+1} = 0$ so that $v_{n+2} + (1 - C(n))\, v_n = 0$, hence v_n and v_{n+2} have unequal sign. In both cases, we see that $\{v_n\}$ oscillates. On the other hand, if all solutions of (10.4) converge then the real solutions of (10.2) do not oscillate. This follows from the fact that all solutions $\{z_n\}$ of (10.4) must necessarily converge to 0. Hence $z_n + 1 = v_{n+1}/v_n$ does not change sign for n large enough. Q.E.D

Proof of Theorem 10.1. We consider the associated matrix recurrence $M_n x_n = x_{n+1}$ with M_n given by $M_n = \begin{pmatrix} 2 & C(n)-1 \\ 1 & 0 \end{pmatrix}$. Then set

$$G_n = \begin{pmatrix} \sqrt{C(n)} & \sqrt{C(n)} \\ -1 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then the matrix recurrence $G_{n+1}^{-1} A^{-1} M_n A G_n y_n = y_{n+1}$ $(n \in \mathbb{N})$ has solutions

$$y_n = \frac{1}{\sqrt{C(n)}} \cdot \begin{pmatrix} v_{n+1} - v_n(1 + \sqrt{C(n)}) \\ v_{n+1} - v_n(1 - \sqrt{C(n)}) \end{pmatrix}$$

for $\{v_n\}$ solutions of (10.2). It follows that the corresponding recurrence of Möbius-transformations

$$z_{n+1} = F_n(z_n) = \frac{c_n z + b_n}{c_n b_n z + 1} \qquad (n \in \mathbb{N}), \tag{10.5}$$

where

$$c_n = \frac{1 - \sqrt{C(n)}}{1 + \sqrt{C(n)}} \qquad \text{and} \qquad b_n = -\frac{\sqrt{C(n+1)/C(n)} - 1}{\sqrt{C(n+1)/C(n)} + 1}$$

has solutions $\{z_n\}$ where

$$z_n = \frac{(v_{n+1}/v_n - 1) \cdot (1/\sqrt{C(n)}) - 1}{(v_{n+1}/v_n - 1) \cdot (1/\sqrt{C(n)}) + 1}. \tag{10.6}$$

We apply Theorem 7.1 to (10.5). Note that $(1 - c_n)/\sqrt{C(n)} \to 2$ and $b_n = -j_1/4n + O(n^{-1-\delta})$, for some number $\delta > 0$. Hence, if $n^2 C(n) \to \infty$, the fixpoints of $F_n(z_n)$, which are the roots of the polynomials $b_n c_n X^2 + (1 - c_n) X - b_n$, tend to 0 and ∞. They are also of bounded variation, since b_n and c_n are, like $C(n)$, finite sums of (fractional) powers of $1/n$ plus some part that is $O(n^{-2-\varepsilon})$, and so are the fixpoints themselves. If ζ_n is the fixpoint of F_n that converges to zero, we have

$$F'_n(\zeta_n) = \frac{1 - s_n}{1 + s_n}, \qquad \text{where} \quad s_n = \sqrt{\left(\frac{1 - c_n}{1 + c_n}\right)^2 + \frac{4 c_n b_n^2}{(1 + c_n)^2}}.$$

This can be seen most easily if we realize that $F'_n(\zeta_n)$ is a quotient of the two eigenvalues of the matrix $\left(\begin{smallmatrix} c_n & b_n \\ c_n b_n & 1 \end{smallmatrix}\right)$ corresponding to F_n (see the final paragraph of Section 2). Thus, by Theorem 7.1, (10.2) is of hyperbolic type, with all solutions of (10.5) converging to one fixpoint, and the remaining solution converging to the other fixpoint, precisely if $\prod_{n=0}^{\infty} |F'_n(\zeta_n)| = 0$ or ∞, hence if $\sum_{n=0}^{\infty} \Re s_n = \pm\infty$, by $|F'_n(\zeta_n)|^2 = 1 - (4\Re s_n / |1 + s_n|^2)$. Relation (10.2) is of elliptic type if $|F'_n(\zeta_n)|$ converges, hence if $\sum_{n=0}^{\infty} |\Re s_n|$ converges (case (3) of Theorem 7.1). In that case, two of the solutions of (10.5) converge to 0 and ∞, whereas the other solutions converge to circles $\{z \in \mathbb{C}: |z| = c\}$.

Furthermore, if $n^2 C(n) \to \infty$ as $n \to \infty$, then $j_1 > -2$ and

$$\begin{aligned} s_n &= \sqrt{C(n) + j_1^2/16n^2 + O(n^{-2-\varepsilon})} \\ &= \sqrt{C(n)}(1 + O(n^{-2-j_1})) \\ &= \sqrt{C(n)} + O(n^{-2-j_1/2}), \end{aligned}$$

hence $\sum_{n=0}^{\infty} \Re s_n$ converges (diverges) precisely when $\sum_{n=0}^{\infty} \Re \sqrt{C(n)}$ does so. Finally, (10.3) follows from (10.6) and the fact that there are (at least) two solutions of (10.5) that converge to 0 and ∞. Let $\{w_n\}$ be any solution of (10.2). Then, by Lemma 10.2, for some constant C,

$$\frac{(v_{n+1}^{(1)}/v_n^{(1)}-1)\cdot(1/\sqrt{C(n)})-(w_{n+1}/w_n-1)\cdot(1/\sqrt{C(n)})}{(v_{n+1}^{(2)}/v_n^{(2)}-1)\cdot(1/\sqrt{C(n)})-(w_{n+1}/w_n-1)\cdot(1/\sqrt{C(n)})}$$
$$= C\cdot\frac{v_n^{(2)}}{v_n^{(1)}} \qquad (n\in\mathbb{N}). \tag{10.7}$$

Thus we see that in the hyperbolic case, where $(w_{n+1}/w_n-1)\cdot(1/\sqrt{C(n)})$ converges to 1 for $\{w_n\}$ linearly independent with $\{v_n^{(1)}\}$, $\lim_{n\to\infty} v_n^{(1)}/v_n^{(2)} = 0$. This follows also from the fact that $\prod_{n=0}^{\infty} |(1-\sqrt{C(n)})/(1+\sqrt{C(n)})| = 0$ if $\sum_{n=0}^{\infty} \Re\sqrt{C(n)} = \infty$. In the elliptic case, for $\{w_n\}$ not a scalar multiple of either of the $\{v_n^{(i)}\}$, the left-hand side of (10.7) does not converge to a point, but to a circle $\{z\in\mathbb{C}\colon |z| = C'\}$, hence the same is true for the right-hand side. We can choose $\{v_n^{(1)}\}$ such that $C=C'$.

Similarly, if $n^2C(n) = d+o(1)$, then $j_1 = -2$ and

$$s_n = \sqrt{\frac{d+1/4}{n^2}(1+O(n^{-\varepsilon}))} = \sqrt{d+1/4}\cdot n^{-1} + O(1/n^{-1-\varepsilon}),$$

hence we are in the elliptic case if $d < -1/4$ and in the hyperbolic case for all other d, except for $d=-1/4$, in which case the fixpoints of (10.5) converge to the same point, so that Theorem 7.1 cannot be applied (we recall that the limits f_1, f_2 of the fixpoints of (10.5) are the zeros of the polynomial $X^2+4\sqrt{d}\,X-1$). By Theorem 7.1, we have solutions $\{v_n^{(1)}\}$, $\{v_n^{(2)}\}$ such that

$$\lim_{n\to\infty}\frac{(v_{n+1}^{(i)}/v_n^{(i)}-1)\cdot(1/\sqrt{C(n)})-1}{(v_{n+1}^{(i)}/v_n^{(i)}-1)\cdot(1/\sqrt{C(n)})+1} = f_i \qquad (i=1,2).$$

It follows that

$$\lim_{n\to\infty}\left(\frac{v_{n+1}^{(i)}}{v_n^{(i)}}-1\right)\cdot n = \sqrt{d}\cdot\frac{1+f_i}{1-f_i} = r_i \qquad (i=1,2),$$

r_1, r_2 being the roots of X^2-X-d.

The remaining assertions concerning the case that $\lim_{n\to\infty} n^2 C(n) = d$ must be treated separately. We consider the matrix recurrence

$$x_{n+1} = A^{-1} M_n A x_n = \begin{pmatrix} 1 & C(n) \\ 1 & 1 \end{pmatrix} x_n \qquad (n \in \mathbb{N}) \tag{10.8}$$

with M_n, A as above, and which has solutions $x_n = \binom{v_{n+1} - v_n}{v_n}$ for $\{v_n\}$ solutions of (10.2). First let $d \neq -1/4$. It follows from a straightforward computation that for

$$H_n = \begin{pmatrix} g_n^{(1)} & g_n^{(2)} \\ 1 & 1 \end{pmatrix}, \qquad g_n^{(i)} = r_i/n \qquad (i = 1, 2; n > 0)$$

with r_1, r_2 the roots of the polynomial $X^2 - X - d$, we have

$$\begin{pmatrix} 1 & d/n(n+1) \\ 1 & 1 \end{pmatrix} H_n = H_{n+1} \cdot \operatorname{diag}\left(1 + \frac{r_1}{n}, 1 + \frac{r_2}{n}\right) \qquad (n > 0).$$

Hence, if $\sum_{n=1}^{\infty} |nC(n) - d/n| < \infty$, then, by $\|H_n\| = O(1)$, $\|H_n^{-1}\| = O((\det H_n)^{-1}) = O(n)$,

$$H_{n+1}^{-1} A^{-1} M_n A H_n = \operatorname{diag}\left(1 + \frac{r_1}{n}, 1 + \frac{r_2}{n}\right) + D_n \qquad (n > 0).$$

with $\|D_n\| = n \cdot O(C(n) - d/n(n+1))$. Applying Lemma 7.2 we find a sequence $\{J_n\}$ converging to the identity matrix I such that

$$J_{n+1}^{-1} H_{n+1}^{-1} A^{-1} M_n A H_n J_n = \operatorname{diag}\left(1 + \frac{r_1}{n}, 1 + \frac{r_2}{n}\right).$$

Let the solutions $\{x_n^{(i)}\}$ of (10.8) be defined by $x_n^{(i)} = H_n J_n(e_i)$ $(i = 1, 2)$. Then

$$x_n^{(i)} = \begin{pmatrix} \hat{v}_{n+1}^{(i)} - \hat{v}_n^{(i)} \\ \hat{v}_n^{(i)} \end{pmatrix} = \begin{pmatrix} r_i/n(1 + o(1)) \\ 1 + o(1) \end{pmatrix} \cdot \prod_{h=1}^{n-1} (1 + r_i/h).$$

Hence, $\hat{v}_n^{(i)}/n^{r_i}$ converges to non-zero numbers c_i $(i = 1, 2)$. Setting $v_n^{(i)} = \hat{v}_n^{(i)}/c_i$ for $i = 1, 2$ we find the desired result.

We proceed similarly for the case that $d = -1/4$. In this case, as is shown by a straightforward computation, we have

$$\begin{pmatrix} 1 & d/n(n+1) \\ 1 & 1 \end{pmatrix} H_n = H_{n+1} \cdot \operatorname{diag}(1 + g_n^{(1)}, 1 + g_n^{(2)}) \qquad (n > 0),$$

where now

$$H_n = \begin{pmatrix} g_n^{(1)} & g_n^{(2)} \\ 1 & 1 \end{pmatrix}, \qquad g_n^{(1)} = \frac{1}{2n},$$

$$g_n^{(2)} = \frac{1}{2n} + \frac{1}{n}\left(\sum_{h=1}^{n-1} \frac{1}{k+1/2}\right)^{-1} \qquad (n>0).$$

In this case, $\|H_n\| = O(1)$, $\|H_n^{-1}\| = O((\det H_n)^{-1}) = O(n \log n)$. Hence, if $\sum_{n=1}^{\infty} n \log n \,|C(n) + 1/(4n^2)| < \infty$, then we can, by Lemma 7.2, find a sequence of matrices $\{J_n\}$ converging to identity and such that

$$J_{n+1}^{-1} H_{n+1}^{-1} A^{-1} M_n A H_n J_n = \operatorname{diag}(1 + g_n^{(1)}, 1 + g_n^{(2)}) \qquad (n>0).$$

The final part of the proof is similar to the case that $d \neq -1/4$.

We show that if $C(n) \in \mathbb{R}$, in the elliptic case the real solutions oscillate. Here we use Lemma 10.3, together with the fact that (10.2) has a subdominant solution if and only if $\lim_{n\to\infty} u_n/v_n$ exists (or is ∞) for all non-zero solutions $\{u_n\}$, $\{v_n\}$ of (10.2). However, in the elliptic case, as we have seen, $\lim_{n\to\infty} (v_n^{(1)}/v_n^{(2)})$ does not converge. Hence, by Lemma 10.3, the real solutions of the recurrence (10.4) diverge and the real solutions of (10.2) oscillate. Q.E.D

The case that $\sum_{n=0}^{\infty} |C(n)|$ converges was already treated by Coffmann [1].

The requirement that if $n^2 C(n) \to d = -1/4$, then $\sum_{n=1}^{\infty} n \log n \,|C(n) + 1/(4n^2)|$ must converge, instead of $\sum_{n=1}^{\infty} n \,|C(n) + 1/(4n^2)|$, as is the case for $d \neq -1/4$, is really necessary. This can be seen from the following fact (Corollary 5.3 of [7]):

THEOREM 10.4. *Consider the recurrence*

$$z_{n+1} = \frac{z_n + C(n)}{1 + z_n} \qquad (n \geqslant n_0) \tag{10.9}$$

for $C(n) \in \mathbb{R}$, *and suppose that* $\lim_{n\to\infty} C(n) = C$ *exists. Set* $\log_j n = \log(\log_{j-1} n)$, $\log_1 n = \log n$. *Then, if there is some number* $\varepsilon > 0$ *such that*

$$C(n) \geqslant -\tfrac{1}{4} \sum_{j=0}^{J} (n \log n \cdots \log_j n)^{-2} + \varepsilon (n \log n \cdots \log_J n)^{-2},$$

then all real solutions of (10.9) *converge, whereas if*

$$C(n) \leqslant -\tfrac{1}{4} \sum_{j=0}^{J} (n \log n \cdots \log_j n)^{-2} - \varepsilon (n \log n \cdots \log_J n)^{-2},$$

then all real solutions of (10.9) *diverge.*

In particular, if $C(n) = -1/4n^2 + c/n^2 \log^2 n$, then the solutions of (10.9) converge if $c \geqslant -1/4$, and diverge for $c < -1/4$. On the other hand, as we have seen in the proof of Theorem 10.1, if $\{v_n\}$ is a solution of (10.2), then $\{z_n\} = \{v_{n+1}/v_n - 1\}$ is a solution of (10.9) (by Lemma 10.3), and the real solutions of (10.9) converge in the hyperbolic case, and diverge in the elliptic case. Notice that Theorem 10.4 also shows that if the $C(n)$ are real, then the real solutions of (10.2) oscillate in the elliptic case.

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